

Eternal forced mean curvature flows III - Morse homology.

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Abstract: We complete the theoretical framework required for the construction of a Morse homology theory for certain types of forced mean curvature flows. The main result of this paper describes the asymptotic behaviour of these flows as the forcing term tends to infinity in a certain manner. This result allows the Morse homology to be explicitly calculated, and will permit us to show in forthcoming work that, for a large family of smooth positive functions, F , defined over a $(d+1)$ -dimensional flat torus, there exist at least 2^{d+1} distinct, locally strictly convex, Alexandrov-embedded hyperspheres of mean curvature prescribed at every point by F .

Key Words: Morse homology, mean curvature, forced mean curvature flow

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1 - Introduction.

1.1 - Prescribed curvature problems. The problem of constructing hypersurfaces of constant curvature subject to geometric or topological restrictions is a standard one of riemannian geometry. The related problem of constructing hypersurfaces whose curvature is *prescribed* by some function of the ambient space is essentially complementary, in that a full understanding of the one generally entails a full understanding of the other. However, as they are usually more straightforward, prescribed curvature problems often serve as a better testing ground for the development of new ideas, as will be the case here.

Consider, therefore, the problem of constructing immersed hyperspheres of prescribed mean curvature inside a $(d+1)$ -dimensional riemannian manifold, M .^{*} It is useful at this stage to introduce some terminology. Thus, S^d will denote the unit sphere inside \mathbb{R}^{d+1} . A smooth immersion, $e : S^d \rightarrow M$, will be said to be an **Alexandrov-embedding** whenever it extends to a smooth immersion, $\tilde{e} : \overline{B}^{d+1} \rightarrow M$, where \overline{B}^{d+1} here denotes the closed unit ball inside \mathbb{R}^{d+1} . $\hat{\mathcal{E}}(M)$ will denote the set of all smooth Alexandrov-embeddings from S^d into M , and $\mathcal{E}(M)$ will denote its quotient under the action by reparametrisation of the group of orientation-preserving, smooth diffeomorphisms of S^d . The spaces $\hat{\mathcal{E}}(M)$ and $\mathcal{E}(M)$ will be furnished respectively with the topology of C^k -convergence for all k , and its induced quotient topology, making $\mathcal{E}(M)$, in particular, into a weakly smooth manifold (c.f. Appendix A). Finally, an element, $[e]$, of $\mathcal{E}(M)$ will be referred to simply as an **Alexandrov-embedding**, and will be identified, at times with a representative element, e , in $\hat{\mathcal{E}}(M)$, and at times with its image in M .

For the purposes of this paper, we may suppose that the extension of any Alexandrov-embedding is actually unique up to diffeomorphism.[†] Thus, for a smooth function, $F : M \rightarrow \mathbb{R}$, the functional, $\mathcal{F} : \mathcal{E}(M) \rightarrow \mathbb{R}$, will be defined by

$$\mathcal{F}([e]) := \text{Vol}([e]) - d \int_{\overline{B}^{d+1}} (F \circ \tilde{e}) d\text{Vol}_{\tilde{e}}, \quad (1)$$

where $\tilde{e} : \overline{B}^{d+1} \rightarrow M$ here denotes the extension of e , and $d\text{Vol}_{\tilde{e}}$ denotes the volume form that it induces over the closed, unit ball. The integral on the right-hand side will be viewed as a weighted volume of the interior of e , so that the functional, \mathcal{F} , will be loosely referred to as the “**area-minus-volume**” functional. The critical points of this functional are precisely those Alexandrov embedded hyperspheres whose mean curvatures are **prescribed** at every point by the function, F ; that is, those elements, $[e]$, of $\mathcal{E}(M)$ such that

$$H_e - F \circ e = 0, \quad (2)$$

^{*} The convention will be adopted throughout this text that the mean curvature of an immersed hypersurface is equal to the *arithmetic mean* of its principal curvatures, as opposed to their sum.

[†] Although this would seem to be the case whenever the ambient manifold is not homeomorphic to a sphere, we know of no such a result in the literature. In the present paper, we will only be concerned with locally strictly convex Alexandrov-embeddings in flat tori. Since all such maps lift to embeddings bounding convex sets in Euclidean space (c.f. [2]), uniqueness of the extension readily follows in this case.

where H_e here denotes the mean curvature of the Alexandrov-embedding, e .

The fact that they arise as critical points suggests that Alexandrov-embedded hyperspheres of prescribed mean curvature should be amenable to study by differential-topological techniques, which should then yield information about their number, at least for generic data. This idea, which is far from new, has already been used by numerous authors to obtain some quite inspiring results in this, and related, settings (c.f., for example [3], [6], [16], [21], [23] and [24]). Of particular relevance to the current discussion, however, is our own, relatively straightforward, result, [14], which shows that, under suitable circumstances, the number of critical points of \mathcal{F} is bounded below by the Euler characteristic of the ambient space. Although this yields existence in certain cases, we have found it rather unsatisfactory, as, on the one hand, it provides little new information in the case where the Euler characteristic of the ambient space vanishes - for example when the ambient space is 3-dimensional - and, on the other, even when this Euler characteristic does not vanish, the number of solutions it yields generally falls short of what Morse theory would lead us to expect.

Although Morse homology theory would appear to be the natural approach for obtaining the best possible lower bounds on the number of critical points of the functional, \mathcal{F} , its development in the current setting has proven to be far from trivial, largely due to the technical challenges involved in making any progress in the theory of mean curvature flows. However, the remarkable non-collapsing theorems recently obtained by Ben Andrews et al. (c.f., for example, [4] and [5]) finally allow the construction of a complete Morse homology theory, at least in the case where the ambient manifold, M , is a flat, $(d + 1)$ -dimensional torus, and where certain further restrictions are also imposed on elements of $\mathcal{E}(M)$.

In [19] and [20], we initiated a programme for the study of the Morse homology of the functional, \mathcal{F} . The objective of the current paper is partly to improve on these results, but mainly to complete the final theoretical step required to complete the construction (c.f. Theorem 1.3.5, below). The remaining work involved, which, though long and technical, is essentially formal, will be completed in our forthcoming paper, [10]. There we will prove the following result, which we already state here in order to clearly illustrate our motivations.

Theorem 1.1.1, In preparation.

If $d \geq 2$ and if T^{d+1} is a $(d + 1)$ -dimensional torus, then, for generic, smooth functions, $F : T^{d+1} \rightarrow]0, \infty[$, such that

$$\sup_{x \in T^{d+1}, \|\xi\|=1} |D^2 F(x)(\xi, \xi)| < (3 - 2\sqrt{2}) \inf_{x \in T^{d+1}} F(x)^3,$$

there exist at least 2^{d+1} distinct Alexandrov-embeddings, $[e] \in \mathcal{E}(T^{d+1})$, of mean curvature prescribed at every point by F .

Remark: Here a $(d + 1)$ -dimensional torus is taken to be any quotient of \mathbb{R}^{d+1} by a cocompact lattice.

Remark: Intriguingly, Theorem 1.3.1, below, would suggest that the Morse homology itself exhibits some sort of bifurcation behaviour as the forcing term moves beyond being

subcritical. However, a deeper understanding of this lies far beyond the scope of the current paper.

Remark: An analogous result should also hold, with suitably modified conditions on F , when the ambient space is 2-dimensional, and also when it is a compact, hyperbolic manifold.

1.2 - Morse homology. In the infinite-dimensional setting, Morse homology algebraically encodes the relationship between the moduli spaces of solutions of certain non-linear parabolic operators, and the moduli spaces of solutions of those elliptic operators which correspond to their stationary states. However, in order to gain some intuition, it is worthwhile to first review the finite-dimensional theory (c.f. [17] for a complete and thorough exposition). Thus, a smooth function, $f : M \rightarrow \mathbb{R}$, will be said to be of **Morse** type whenever every one of its critical points is non-degenerate. It will then be said to be of Morse-Smale type whenever, in addition, every one of its complete gradient flows is non-degenerate in the following sense. Recall that a complete gradient flow, $\gamma : \mathbb{R} \rightarrow M$, of f is, by definition, a zero of the non-linear differential operator

$$\gamma \mapsto D_t \gamma + \nabla f(\gamma(t)). \quad (3)$$

The linearisation of this operator about γ , which maps sections of the pull-back bundle, γ^*TM , to other sections of the same bundle, is given by

$$\xi \mapsto \nabla_{\partial_t} \xi + \text{Hess}(f)(\gamma(t))\xi. \quad (4)$$

The Morse property of f ensures that this operator is always of Fredholm type, and the function, f , will then be of **Morse-Smale** type whenever this operator is surjective for all γ . There is no shortage of functions with this property. Indeed, standard transversality results show that the set of all such functions is generic, that is, of the second category in the sense of Baire. This means that it contains the intersection of a countable family of open, dense subsets of $C^\infty(M)$, so that, in particular, by the Baire category theorem, it is dense.

By the Morse property of f , every one of its critical points is isolated, so that the set, Z , of all its critical points is finite. In particular, for all k , the subset, Z_k , of Z , defined to be the set of critical points of Morse index k , is also finite, where the **Morse index** of a critical point is here defined to be the sum of the multiplicities of all strictly negative eigenvalues of $\text{Hess}(f)$ at that point. For all k , the k 'th order chain group of the Morse homology of (M, f) will then be defined by

$$C_k(M, f) := \mathbb{Z}_2[Z_k].$$

Equivalently, $C_k(M, f)$ will be the \mathbb{Z}_2 -module of all \mathbb{Z}_2 -valued functions over Z_k . In particular, the sum of the dimensions of the chain groups is equal to the cardinality of the solution set, Z .

We will denote by W the space of all complete gradient flows of f , furnished with the topology of C_{loc}^k -convergence for all k . By compactness and the Morse property of f , every

complete gradient flow, γ , has well-defined end-points, p and q , in the sense that

$$\begin{aligned} \lim_{t \rightarrow -\infty} \gamma(t) &= p, \text{ and} \\ \lim_{t \rightarrow +\infty} \gamma(t) &= q. \end{aligned}$$

Furthermore, these end-points are always critical points of f , so that

$$W = \bigcup_{(p,q) \in Z \times Z} W_{p,q},$$

where $W_{p,q}$ here denotes the space of all complete gradient flows of f starting at p and ending at q . The Morse-Smale property of f ensures that, for all p, q , $W_{p,q}$ in fact has the structure of a smooth manifold of dimension equal to the difference between the respective Morse indices of p and q . Of particular interest is the case where this difference is equal to 1, and where the space $W_{p,q}$ is therefore a one-dimensional manifold. Indeed, since the differential operator (3) defining elements of $W_{p,q}$ is homogeneous in time, the additive group, \mathbb{R} , acts on this space by translation of the time variable, yielding a quotient space which happens to be a compact, zero-dimensional manifold: that is, a *finite set of points*. The cardinality of this set will then be used to define the boundary operator of the chain complex of (M, f) . Indeed, for all $p \in Z_k$,

$$\partial_k p := \sum_{q \in Z_{k-1}} [\#W_{p,q}/\mathbb{R}] q.$$

The first main theorem of Morse homology theory states that, for all k , the composition, $\partial_{k-1} \circ \partial_k$, vanishes. It follows that the chain complex, $(C_*(M), \partial_*)$, has a well-defined homology,

$$H_*(M, f) := \text{Ker}(\partial_*) / \text{Im}(\partial_{*+1}),$$

and this will be called the **Morse homology** of (M, f) . The second and third main theorems of Morse homology theory then state respectively that the Morse homology is, up to isomorphism, independent of the Morse-Smale function used, and, furthermore, that it is in fact isomorphic to the singular homology of the ambient space, M . Significantly, since the sum of the dimensions of the homology groups yields a lower bound for the sum of the dimensions of the chain groups, it also yields a lower bound for the cardinality of the solution set, Z . In this manner, it is shown that the number of critical points of a generic function, f , is bounded below by the sum of the Betti numbers of the ambient manifold, which is one of the main results of Morse theory.

1.3 - Eternal forced mean curvature flows. Given a riemannian manifold, M , and a smooth, positive function, $F : M \rightarrow]0, \infty[$, an **eternal forced mean curvature flow** with forcing term F will be a strongly smooth* curve $[e_t] : \mathbb{R} \rightarrow \mathcal{E}(M)$ such that

$$\langle \partial_t e_t, N_t \rangle + H_t - F \circ e_t = 0, \tag{5}$$

* c.f. Appendix A for terminology.

where N_t is the outward-pointing, unit, normal vector field of the embedding, e_t , and H_t is its mean curvature. The eternal forced mean curvature flows with forcing term F are precisely the L^2 -gradient flows of the “area-minus-volume” functional, \mathcal{F} , introduced in Section 1.1. Thus, bearing in mind the discussion of the preceding section, a Morse homology theory for the pair $(\mathcal{E}(M), \mathcal{F})$ should follow once the appropriate properties of eternal forced mean curvature flows have been established. The objective of the current paper will be to obtain precisely these properties in the case where M is a $(d+1)$ -dimensional torus, T^{d+1} .[†] We will first refine what has already been proven in [19] and [20], before stating the new results which complete the theoretical side of the construction. Observe, in particular, that it will be sufficient to obtain results for solutions of the *parabolic problem*, that is, for eternal forced mean curvature flows, since solutions of the *elliptic problem*, that is, Alexandrov-embedded hyperspheres of prescribed curvature, arise as special cases of the former, being merely those forced mean curvature flows which are constant.

It will first be necessary to introduce geometric restrictions. Thus, for all $\lambda \geq 1$, the subspace $\mathcal{E}_\lambda(T^{d+1})$ of $\mathcal{E}(T^{d+1})$ will be identified as follows. First, an embedding, $[e]$, will be said to be **locally strictly convex** (or LSC) whenever every one of its principal curvatures is strictly positive. An LSC Alexandrov-embedding, $[e]$, will then be said to be **pointwise λ -pinched** whenever it has the property that, for every point $x \in S^d$,

$$\kappa_d(x) \leq \lambda \kappa_1(x),$$

where $0 < \kappa_1(x) \leq \kappa_d(x)$ are, respectively, the least and greatest principal curvatures of the Alexandrov-embedding, e , at the point x . Next recall that any LSC Alexandrov-embedding in \mathbb{R}^{d+1} is actually the boundary of some open, convex set (c.f. [2]). A pointwise λ -pinched Alexandrov-embedding, $[e]$, in \mathbb{R}^{d+1} will then be said to be λ -non-collapsed whenever it has the property that for all $x \in S^d$, the Euclidean hypersphere of curvature $\lambda \kappa_1(x)$ which is an interior tangent to $[e]$ at the point x is entirely contained within the closed set bounded by $[e]$.^{*} A pointwise λ -pinched Alexandrov-embedding, $[e]$, in T^{d+1} is then said to be **λ -non-collapsed** whenever its lift to \mathbb{R}^{d+1} has this property. Finally, for all $\lambda \geq 1$, the subspace $\mathcal{E}_\lambda(T^{d+1})$ of $\mathcal{E}(T^{d+1})$ will be defined to be the set of all Alexandrov-embeddings, $[e]$, which are LSC, pointwise λ -pinched, and λ -non-collapsed.

Consider now the rational function $\phi : [1, \infty[\rightarrow [0, \infty[$ given by

$$\phi(t) := \frac{(t-1)}{t(t+1)}. \quad (6)$$

Observe that this function tends to 0 at 1 and $+\infty$ and has a unique maximum, equal to

[†] Throughout this text, a $(d+1)$ -dimensional torus will be taken to be *any* quotient of \mathbb{R}^{d+1} by a cocompact lattice.

^{*} We alert the reader to the fact that our notion of non-collapsedness compares the curvature of the Euclidean hypersphere to the *least* principal curvature of the Alexandrov-embedding. Although this may be at first surprising, it makes good sense, since pointwise λ -pinching ensures that the curvature of the Euclidean hypersphere is actually no less than the greatest principal curvature of the Alexandrov-embedding.

$(3 - 2\sqrt{2})$, at the point $(1 + \sqrt{2})$. There therefore exist two smooth inverses,

$$\lambda : [0, (3 - 2\sqrt{2})[\rightarrow [1, 1 + \sqrt{2}[, \text{ and}$$

$$\Lambda : [0, (3 - 2\sqrt{2})[\rightarrow]1 + \sqrt{2}, +\infty].$$

The forcing term, F , will be said to be **sub-critical** whenever it is strictly positive, and

$$\sup_{x \in T^{d+1}, \|\xi\|=1} |D^2(F)(x)(\xi, \xi)| \leq (3 - 2\sqrt{2}) \inf_{x \in T^{d+1}} F(x)^3. \quad (7)$$

In this case, the constants $\lambda_F < \Lambda_F$ will be defined by

$$\begin{aligned} \lambda_F &:= \lambda \left(\sup_{x \in T^{d+1}, \|\xi\|=1} |D^2(F)(x)(\xi, \xi)| / \inf_{x \in T^{d+1}} F(x)^3 \right), \text{ and} \\ \Lambda_F &:= \Lambda \left(\sup_{x \in T^{d+1}, \|\xi\|=1} |D^2(F)(x)(\xi, \xi)| / \inf_{x \in T^{d+1}} F(x)^3 \right), \end{aligned} \quad (8)$$

and for subcritical F , the subspace $\mathcal{E}_F(T^{d+1})$ will be defined by

$$\mathcal{E}_F(T^{d+1}) := \mathcal{E}_{\Lambda_F}(T^{d+1}). \quad (9)$$

Embeddings in $\mathcal{E}_F(T^{d+1})$ will be called **admissible**. Likewise, a forced mean curvature flow, $[e_t]$, with forcing term, F , will be said to be **admissible** whenever $[e_t]$ is admissible for all t . Morse homology will be constructed for the pair $(\mathcal{E}_F(T^{d+1}), \mathcal{F})$.

Significantly, the space, $\mathcal{E}_F(T^{d+1})$, of admissible Alexandrov-embeddings is actually a *closed* subset of $\mathcal{E}(T^{d+1})$. However, it is of fundamental importance in any differential topology theory that the space of admissible elements be *open*. Indeed, otherwise, the perturbative stages of the construction would cease to function. This problem of openness is typically addressed indirectly in the statement of the compactness result, which is the case in our earlier work, [19], where we prove compactness for families of eternal forced mean curvature flows, and where the rather restrictive conditions imposed actually serve to ensure that all limits remain within a given open set. The greater generality of the present setting is obtained via the following result, where the problem of openness is addressed via an adaptation of Ben Andrews' non-collapsing theorem. Indeed, even though the subset, $\mathcal{E}_F(T^{d+1})$, is not open, for subcritical F , the Morse homology of $(\mathcal{E}_F(T^{d+1}), \mathcal{F})$ lies strictly in its interior, and is, in particular, separated from the rest of the Morse homology of $(\mathcal{E}(T^{d+1}), \mathcal{F})$. An eternal forced mean curvature flow $[e_t] : \mathbb{R} \rightarrow \mathcal{E}(T^{d+1})$ will be said to be of **bounded type** whenever

$$\sup_t \text{Diam}([e_t]) < \infty.$$

In Section 3, we prove

Theorem 1.3.1, Separation.

If $[e_t] : \mathbb{R} \rightarrow \mathcal{E}_F(T^{d+1})$ is an eternal forced mean curvature flow of bounded type with sub-critical forcing term F , and if $[e_t]$ is pointwise Λ_F -pinched and Λ_F -non-collapsed for all t , then $[e_t]$ is pointwise λ_F -pinched and λ_F -non-collapsed for all t .

Theorem 1.3.1, justifies the context of all that follows. First, the following compactness result is obtained via a straightforward blow-up argument (c.f. Section 2.2).

Theorem 1.3.2, Compactness.

Fix $d \geq 2$, and let (F_m) be a sequence of smooth, positive functions over \mathbb{R}^{d+1} converging in the C_{loc}^k -sense for all k to the smooth, positive function F_∞ . Suppose, furthermore, that

$$0 < \inf_m F_{m,-} \leq \sup_m F_{m,+} < \infty.$$

For all m , let $[e_{m,t}] : \mathbb{R} \rightarrow \mathcal{E}(\mathbb{R}^{d+1})$ be an eternal forced mean curvature flow of bounded type with forcing term F_m . Suppose, furthermore, that there exists $\lambda \geq 1$ such that, for all m , and for all t , $[e_{m,t}]$ is pointwise λ -pinched and λ -non-collapsed. If there exists a compact subset $K \subseteq \mathbb{R}^{d+1}$ such that $[e_{m,0}] \cap K \neq \emptyset$ for all m , then there exists an eternal forced mean curvature flow, $[e_{\infty,t}]$, towards which the sequence $([e_{m,t}])$ subconverges in the C_{loc}^k -sense for all k . In particular, $[e_{\infty,t}]$ is also of bounded type and, for all t , $[e_{\infty,t}]$ is pointwise λ -pinched and λ -non-collapsed.

As in the finite-dimensional case, the functional, \mathcal{F} , will be said to be of **Morse type** whenever every one of its admissible critical points is non-degenerate. Recalling that Theorem 1.3.2 also applies to sequences of critical points of \mathcal{F} , straightforward differential-topological arguments now show (c.f., for example, [14] and [23]) that the functional, \mathcal{F} , has this property for generic F . In this case, it readily follows from Theorem 1.3.2, again, that eternal forced mean curvature flows always have well-defined end-points.

Theorem 1.3.3, End-points.

If \mathcal{F} is of Morse type, and if $[e_t] : \mathbb{R} \rightarrow \mathcal{E}_F(T^{d+1})$ is an admissible eternal forced mean curvature flow of bounded type, then there exist admissible embeddings, $[e_\pm] \in \mathcal{E}_F(T^{d+1})$, of mean curvature prescribed by F such that

$$\begin{aligned} \lim_{t \rightarrow -\infty} [e_t] &= [e_-], \text{ and} \\ \lim_{t \rightarrow +\infty} [e_t] &= [e_+]. \end{aligned}$$

However, the main result of this paper actually concerns the asymptotic behaviour of the Morse homology of $(\mathcal{E}_F(T^{d+1}), \mathcal{F})$ as F tends to infinity. Indeed, it is this that will yield an explicit formula for the Morse homology. Consider therefore a smooth function, $f : T^{d+1} \rightarrow \mathbb{R}$, of Morse-Smale type. For all sufficiently small $\kappa > 0$, consider the forcing term

$$F_\kappa := \frac{1}{\kappa}(1 - \kappa^2 f),$$

let \mathcal{F}_κ denote its “area-minus-volume” functional, and let $\mathcal{E}_\kappa(T^{d+1})$ denote the space of all Alexandrov-embedded hyperspheres which are admissible for F_κ , that is

$$\mathcal{E}_\kappa(T^{d+1}) := \mathcal{E}_{F_\kappa}(T^{d+1}).$$

In [18], building on the work, [25], of Ye, we show (c.f. also [12]),

Theorem 1.3.4, Concentration: elliptic case.

For sufficiently small $\kappa > 0$, there exists a strongly smooth map $\Phi : T^{d+1} \rightarrow \mathcal{E}_\kappa(T^{d+1})$ such that for any critical point, p , of f of Morse index k , the point $\Phi(p)$ is a non-degenerate critical point of \mathcal{F}_κ of Morse index $(k + 1)$. Furthermore, \mathcal{F}_κ has no other critical points in $\mathcal{E}_\kappa(T^{d+1})$.

The Morse homology of $(\mathcal{E}_\kappa(T^{d+1}), \mathcal{F}_\kappa)$ is explicitly determined once the corresponding asymptotic behaviour for eternal forced mean curvature flows has been proven. Thus, given any two critical points, $[e]$ and $[f]$, of \mathcal{F}_κ , $\mathcal{W}_{[e],[f]}$ will denote the space of admissible forced mean curvature flows of bounded type with forcing term F_κ , starting at $[e]$ and ending at $[f]$. Next, an eternal forced mean curvature flow, $[e_t] : \mathbb{R} \rightarrow \mathcal{E}(T^{d+1})$, will be said to be **non-degenerate** whenever its linearised mean curvature flow operator is surjective. If every element of $\mathcal{W}_{[e],[f]}$ is non-degenerate, and if the functional, \mathcal{F}_κ , is of Morse type, then it follows by standard techniques of Fredholm theory that $\mathcal{W}_{[e],[f]}$ has the structure of a smooth, finite-dimensional manifold. In Sections 4 and 5, making use of the compactness result of Theorem 1.3.2, together with the existence of well-defined end-points established in Theorem 1.3.3, we complement Theorem 1.3.4 by showing

Theorem 1.3.5, Concentration: parabolic case.

For sufficiently small $\kappa > 0$, and for every pair of critical points, p and q , of f such that $\text{Index}(p) - \text{Index}(q) = 1$, every element of $\mathcal{W}_{\Phi(p),\Phi(q)}$ is non-degenerate, and there exists a canonical diffeomorphism

$$\hat{\Phi} : W_{p,q} \rightarrow \mathcal{W}_{\Phi(p),\Phi(q)}.$$

In particular, there are no other admissible eternal forced mean curvature flows of bounded type with forcing term F_κ , starting at $\Phi(p)$ and ending at $\Phi(q)$.

Remark: This result follows directly from Theorem 4.4.2 and Lemma 5.4.1, below. In actual fact, in [20], the perturbative part of this result has been shown in the slightly different context of forced mean curvature flows with constant forcing term inside general riemannian manifolds.

1.4 - Constructing the Morse homology. It remains to sketch how the results of Section 1.3 serve to construct the Morse homology of $(\mathcal{E}_F(T^{d+1}), \mathcal{F})$. First, for a subcritical forcing term, F , the **elliptic solution space**, \mathcal{Z} , is defined to be the set of all critical points of \mathcal{F} ; that is, the set of all admissible, Alexandrov-embedded hyperspheres of mean curvature prescribed by F . As indicated above, it follows from the compactness result of Theorem 1.3.2 that, for generic F , the “area-minus-volume” functional, \mathcal{F} , is of Morse type in the sense that every one of its critical points is non-degenerate. In particular, in this case, \mathcal{Z} consists only of isolated elements, and, by the compactness result of Theorem 1.3.2, again, is finite.

It follows from the spectral theory of second-order, elliptic operators that every element, $[e]$, of \mathcal{Z} has finite Morse index, defined here to be the sum of the multiplicities of all the strictly negative eigenvalues of the Jacobi operator of \mathcal{F} at e , (c.f. [14] and [23]). For all integer k , the finite set, \mathcal{Z}_k , is then defined to be the set of all critical points of \mathcal{F} of Morse index k , and, the k 'th order chain group of $(\mathcal{E}_F(T^{d+1}), \mathcal{F})$ is defined by

$$\mathcal{C}_k(\mathcal{E}_F(T^{d+1}), \mathcal{F}) := \mathbb{Z}_2[\mathcal{Z}_k],$$

so that, in particular, the sum of the dimensions of the chain groups is equal to the cardinality of the elliptic solution space, \mathcal{Z} .

The **parabolic solution space**, \mathcal{W} , is defined to be the space of all admissible, eternal forced mean curvature flows of bounded type and with forcing term F furnished with the topology of C_{loc}^k -convergence for all k . By the existence of well-defined end-points established in Theorem 1.3.3, this space decomposes as

$$\mathcal{W} = \bigcup_{[e],[f] \in \mathcal{Z} \times \mathcal{Z}} \mathcal{W}_{[e],[f]},$$

where $\mathcal{W}_{[e],[f]}$ here denotes the subspace of \mathcal{W} consisting of those flows which start at $[e]$ and which end at $[f]$. Upon perturbing \mathcal{F} in a straightforward manner, we may suppose that every element of $\mathcal{W}_{[e],[f]}$ is non-degenerate, so that, by the inverse function theorem, the space, $\mathcal{W}_{[e],[f]}$, carries the structure of a smooth manifold of finite-dimension, which, by the Atiyah-Patodi-Singer Index Theorem (c.f. [13]), is equal to the difference between the respective Morse indices of $[e]$ and $[f]$. In particular, when this difference is equal to 1, $\mathcal{W}_{[e],[f]}$ is 1-dimensional, and since the equation, (5), defining elements of \mathcal{W} is homogeneous in time, the additive group, \mathbb{R} , acts on this space by translation of the time variable, yielding a quotient space which, as before, turns out to be a compact, 0-dimensional manifold: that is, a finite set of points. The cardinality of this set is used to define the boundary operator of the chain complex of $(\mathcal{E}_F(T^{d+1}), \mathcal{F})$. Indeed, for all $[e] \in \mathcal{Z}_k$,

$$\partial_l[e] := \sum_{[f] \in \mathcal{C}_{l-1}} [\#\mathcal{W}_{[e],[f]}/\mathbb{R}] [f].$$

The key result of Morse homology theory is that the square of the boundary operator vanishes. The two main components of this result are a compactness theorem modulo broken trajectories and, conversely, a glueing theorem for broken trajectories. However, a straightforward combinatorial argument, outlined in [19], shows how compactness modulo broken trajectories readily follows from the compactness result already obtained in Theorem 1.3.2. The remaining steps, including the glueing theorem, then follow by general arguments valid for large families of suitably regular parabolic operators. It is therefore a straightforward, though highly technical, matter to prove the fundamental relation

$$\partial_{l-1} \circ \partial_l = 0,$$

for all l . It follows that the chain complex, $\mathcal{C}_*(\mathcal{E}_F(T^{d+1}), \mathcal{F})$, has a well defined homology,

$$\mathcal{H}_*(\mathcal{E}_F(T^{d+1}), \mathcal{F}) := \text{Ker}(\partial_*)/\text{Im}(\partial_{*+1}),$$

which we call the Morse homology of $(\mathcal{E}_F(T^{d+1}), \mathcal{F})$.

Similar arguments then show that, up to isomorphism, the Morse homology is independent of the generic, sub-critical function, F , used, and it is then explicitly calculated by considering the case where $F = F_\kappa$, for some Morse-Smale function, f , and for suitably small κ . Indeed, by Theorem 1.3.4, for all l , and for all sufficiently small κ , there exists a canonical isomorphism

$$\Phi_l : C_l(X, f) \rightarrow \mathcal{C}_{l+1}(\mathcal{E}_\kappa(T^{d+1}), \mathcal{F}_\kappa).$$

Furthermore, by Theorem 1.3.5, upon reducing κ further if necessary,

$$\partial_{l+1} \circ \Phi_l = \Phi_{l-1} \circ \partial_l,$$

so that Φ_l quotients down to another isomorphism

$$\Phi_l : H_l(X, f) \rightarrow \mathcal{H}_{l+1}(\mathcal{E}_\kappa(T^{d+1}), \mathcal{F}_\kappa).$$

Since $H_*(X, f)$ is known to be isomorphic to the singular homology of the torus, it then follows that, for generic, subcritical F ,

$$H_l(\mathcal{E}_F(T^{d+1}), \mathcal{F}) = \begin{cases} \mathbb{Z}_2^{\binom{d+1}{l-1}} & \text{if } 1 \leq l \leq d+2 \\ 0 & \text{otherwise.} \end{cases}$$

Finally, since the sum of the dimensions of the homology groups yields a lower bound for the sum of the dimensions of the chain groups, it also yields a lower bound for the cardinality of the elliptic solution space, \mathcal{Z} , so that, for a generic, subcritical forcing term, F ,

$$\#\mathcal{Z} \geq \sum_{l=1}^{d+2} \binom{d+1}{l-1} = 2^{d+1},$$

thus proving Theorem 1.1.1.

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2 - Compactness and end-points.

2.1 - Geometric bounds. Given a riemannian manifold, M , and a smooth, positive function $F : M \rightarrow]0, \infty[$, denote

$$F_- := \inf_{x \in M} F(x), \text{ and} \\ F_+ := \sup_{x \in M} F(x).$$

Consider now the case where the ambient space is $(d+1)$ -dimensional Euclidean space.

Lemma 2.1.1

If $[e_t] : \mathbb{R} \rightarrow \mathcal{E}(\mathbb{R}^{d+1})$ is an eternal forced mean curvature flow of bounded type with forcing term F , then, for all t , $[e_t]$ contains no Euclidean hypersphere of mean curvature less than F_- .

Proof: Suppose the contrary. Without loss of generality, $[e_0]$ contains the Euclidean hypersphere of mean curvature $a < F_-$ about 0. Let $[f_t] : [0, \infty[\rightarrow \mathcal{E}(\mathbb{R}^{d+1})$ be the forced mean curvature flow with constant forcing term, F_- , starting at this hypersphere. Explicitly,

$$f(t, x) = \psi^{-1} \left(F_-^2 t + \psi \left(\frac{1}{a} \right) \right) x,$$

where $\psi :]1/F_-, \infty[\rightarrow \mathbb{R}$ is given by

$$\psi(r) := (rF_- - 1) + \log(rF_- - 1).$$

Since $[f_0]$ is contained in $[e_0]$, and since $F_- \leq F$, it follows by the geometric maximum principle that $[f_t]$ is contained in $[e_t]$ for all t . However, this is not possible, since $[e_t]$ is of bounded type, but $[f_t]$ is unbounded, and the result follows. \square

Lemma 2.1.2

If $[e_t] : \mathbb{R} \rightarrow \mathcal{E}(\mathbb{R}^{d+1})$ is an eternal forced mean curvature flow with forcing term, F , then, for all t , $[e_t]$ is contained within no Euclidean hypersphere of mean curvature greater than F_+ .

Proof: Suppose the contrary. Without loss of generality, $[e_0]$ is contained within the Euclidean hypersphere of curvature $a > F_+$ about 0. Let $[f_t] : [0, R[\rightarrow \mathcal{E}(\mathbb{R}^{d+1})$ be the forced mean curvature flow with constant forcing term, F_+ , starting at this hypersphere. Explicitly,

$$f(t, x) = \psi^{-1} \left(F_+^2 t + \psi \left(\frac{1}{a} \right) \right) x,$$

where $\psi :]0, 1/F_+[\rightarrow \mathbb{R}$ is given by

$$\psi(r) := (rF_+ - 1) + \log(1 - rF_+).$$

Since $[f_0]$ contains $[e_0]$ and since $F_+ \geq F$, it follows by the geometric maximum principle that $[f_t]$ contains $[e_t]$ for all t . However, this is not possible, since $[e_t]$ exists for all time, but $[f_t]$ extinguishes in finite time, and the result follows. \square

Given a constant, $c > 0$, an LSC embedding, $[e]$ is said to be **c -convex** whenever its least principal curvature is at every point bounded below by c .

Lemma 2.1.3

Let $[e_t] : \mathbb{R} \rightarrow \mathcal{E}(\mathbb{R}^{d+1})$ be an eternal forced mean curvature flow of bounded type with forcing term F . If $[e_t]$ is λ -non-collapsed, then it is also c -convex, where

$$c = \frac{F_-}{\lambda}.$$

Proof: Indeed, fix $t \in \mathbb{R}$ and let $x \in S^d$ minimise κ_1 . Since $[e_t]$ is λ -non-collapsed, the hypersphere of curvature $\lambda\kappa_1(x)$ which is tangent to $[e_t]$ at x is contained within $[e_t]$. Thus, by Lemma 2.1.1, $\lambda\kappa_1(x) > F_-$, and the result follows. \square

In particular, the Bonnet-Meyer Theorem yields explicit bounds on the diameter of each $[e_t]$.

Corollary 2.1.4

With the same hypotheses as in Lemma 2.1.3, for all t ,

$$\text{Diam}([e_t]) \leq \sqrt{\frac{\lambda}{F_-}} \pi.$$

2.2 - Compactness. Consider a sequence, (F_m) , of smooth, positive functions over \mathbb{R}^{d+1} converging in the C_{loc}^k -sense for all k to the smooth, positive function, F_∞ . Suppose, furthermore, that

$$0 < \inf_m F_{m,-} \leq \sup_m F_{m,+} < \infty,$$

where $F_{m,-}$ and $F_{m,+}$ are defined as in the preceding section. For all m , let $[e_{m,t}] : \mathbb{R} \rightarrow \mathcal{E}(\mathbb{R}^{d+1})$ be an eternal forced mean curvature flow of bounded type with forcing term F_m , and suppose that there exists a compact subset, K , of \mathbb{R}^{d+1} such that

$$[e_{m,0}] \cap K \neq \emptyset$$

for all m . Recall first the classical

Lemma 2.2.1, Quasi-maximum lemma.

Let X be a complete metric space. Let $\phi : X \rightarrow]0, \infty[$ be a positive function. For all $x_0 \in X$, for all $C > 1$, and for all $A, \alpha > 0$, there exists x in X such that $\phi(x) \geq \phi(x_0)$, and

$$\phi(y) \leq C\phi(x)$$

for all $y \in B_{A\phi(x)^{-\alpha}}(x)$, where $B_{A\phi(x)^{-\alpha}}(x)$ here denotes the ball of radius $A\phi(x)^{-\alpha}$ about the point, x .

Proof: Indeed, otherwise, by induction, there exists a sequence, (x_m) , of points in X such that, for all m , $x_{m+1} \in B_{A\phi(x)^{-\alpha}}(x_m)$, and $\phi(x_{m+1}) > C\phi(x_m)$. By induction again, for all m , $\phi(x_m) > C^m\phi(x)$, and so $d(x_{m+1}, x_m) < A\phi(x_0)^{-\alpha}C^{-m\alpha}$. Since the series $\sum_{m=0}^{\infty} C^{-m\alpha}$ converges, it follows that (x_m) is a Cauchy sequence in X . By completeness, there therefore exists x_∞ in X towards which (x_m) converges. However, this is absurd, since the sequence $(\phi(x_m))$ diverges, but ϕ is continuous. The result follows. \square

Lemma 2.2.2

If $d \geq 2$, and if, for some $\lambda \geq 1$, $[e_{m,t}]$ is pointwise λ -pinched and λ -non-collapsed for all m , then there exists $B > 0$ such that, for all m , and for all (t, x) ,

$$\kappa_{m,d,t}(x) \leq B,$$

where $\kappa_{m,d,t}$ here denotes the greatest principal curvature of the embedding $e_{m,t}$.

Proof: Suppose the contrary. For all m , let $K_m : \mathbb{R} \rightarrow \mathbb{R}$ be the function given by

$$K_m(t) = \sup_{x \in S^d} \kappa_{m,t,d}(x),$$

and consider a sequence (t_m) , chosen such that $(K_m(t_m))$ tends to $+\infty$. By the quasi-maximum lemma (Lemma 2.2.1), without loss of generality, for all m , and for all $|s - t_m| < K_m(t_m)^{-1/2}$,

$$K_m(s) < 2K_m(t_m).$$

For all m , let x_m be a point of S^d maximising K_d , and define $[\tilde{e}_{m,t}]$ by

$$\tilde{e}_{m,t}(x) := K_m(t_m) (e_{m,(t-t_m)/K_m(t_m)}(x) - e_{m,t_m}(x_m)),$$

so that, for all m , $[\tilde{e}_{m,t}]$ is an eternal forced mean curvature flow of bounded type with forcing term

$$\tilde{F}_{m,t} := \frac{1}{K_m(t_m)} (F_m(x/K_m(t_m) + x_m)).$$

Furthermore, for all m , $\tilde{e}_m(0, x_m) = 0$, $\tilde{\kappa}_{m,d}(0, x_m) = 1$, and

$$\tilde{\kappa}_{m,d}(t, x) \leq 2$$

for all $(t, x) \in [-K_m(t_m)^{1/2}, K_m(t_m)^{1/2}] \times S^d$.

For all m , let X_m be the convex subset bounded by the embedded hypersphere $[\tilde{e}_{m,0}]$. Let X_∞ be a convex subset towards which (X_m) subconverges in the local Hausdorff sense. It follows by the above estimates and hypoelliptic regularity that the sequence, (X_m) , actually subconverges towards X_∞ in the C_{loc}^k -sense for all k . In particular, ∂X_∞ is convex, pointwise λ -pinched and λ -non-collapsed, and its greatest principal curvature at the point x_∞ is equal to 1. In particular, X_∞ is strictly convex, since, otherwise, being λ -non-collapsed, it would coincide with a half-space and therefore have zero curvature at every boundary point, which is absurd. Since $d \geq 2$, it follows by the result, [7], of Hamilton that X_∞ is compact, and there therefore exists $C > 0$ such that, after extracting a subsequence,

$$\text{Diam}([e_{m,t_m}]) \leq C/K_m(t_m)$$

for all m . In particular, for sufficiently large m , $[e_{m,t_m}]$ is contained within a Euclidean hypersphere of arbitrarily large curvature. This is absurd by Lemma 2.1.2, and the result follows. \square

This proves the compactness result for eternal forced mean curvature flows.

Theorem 1.3.2, Compactness

If $d \geq 2$, and if, for some $\lambda \geq 1$, $[e_{m,t}]$ is pointwise λ -pinched and λ -non-collapsed for all m , then there exists an eternal forced mean curvature flow, $[e_{\infty,t}]$, of bounded type with forcing term F_{∞} towards which $[e_{m,t}]$ subconverges in the C_{loc}^k -sense for all k . In particular, $[e_{\infty,t}]$ is also pointwise λ -pinched and λ -non-collapsed.

Proof: By standard arguments from the theory of quasi-linear parabolic PDEs, it suffices to show that the diameters and principal curvatures of all the spheres $([e_{m,t}])$ are uniformly bounded above. The result now follows by Corollary 2.1.4 and Lemma 2.2.2. \square

2.3 - End points. Consider now the case where the ambient manifold is a $(d+1)$ -dimensional torus. Consider a smooth, positive function, $F : T^{d+1} \rightarrow]0, \infty[$ and admissible, eternal forced mean curvature flows in T^{d+1} with forcing term F . First recall

Lemma 2.3.1, Unique continuation.

If $[e_t], [f_t] : \mathbb{R} \rightarrow \mathcal{E}(T^{d+1})$ are eternal forced mean curvature flows with forcing term F , and if $[f_t] = [e_{t+\delta}]$ for some t and for some δ , then $[f_t] = [e_{t+\delta}]$ for all t .

Proof: This follows by a standard argument using the result, [1], of Agmon & Nirenberg (c.f. [22] for details). Observe, in particular, that energy-boundedness is not a concern for us, since the flows considered here are all smooth. \square

Denote by $\mathcal{F}([e_t])$ the set $\{\mathcal{F}([e_t]) \mid t \in \mathbb{R}\}$.

Lemma 2.3.2

Let $[e_t] : \mathbb{R} \rightarrow \mathcal{E}(T^{d+1})$ be an admissible, eternal forced mean curvature flow of bounded type with forcing term F . If $[e_t]$ is non-constant, then the set $\mathcal{F}([e_t])$ is an open, bounded interval.

Proof: Consider the smooth function, $\phi : \mathbb{R} \rightarrow \mathbb{R}$, given by $\phi(t) = \mathcal{F}([e_t])$. In particular, $\mathcal{F}([e_t]) = \text{Im}(\phi)$. Since ϕ is continuous, $\text{Im}(\phi)$ is connected, and is therefore an interval. Since $[e_t]$ is an L^2 -gradient flow of \mathcal{F} , $\phi'(t) = 0$ if and only if $[e_t]$ is a critical point of \mathcal{F} . It follows by the unique continuation result of Lemma 2.3.1, that if ϕ' vanishes at any point, then $[e_t]$ is the constant flow sending \mathbb{R} to a critical point of \mathcal{F} . Since this case is excluded by hypothesis, ϕ' never vanishes, and so $\text{Im}(\phi)$ is open.

It remains to show that $\mathcal{F}([e_t])$ is bounded. Suppose, first, that it is not bounded above. Then there exists a sequence (t_m) (converging to $-\infty$) such that $(\mathcal{F}([e_{t_m}]))$ tends to infinity. For all m , define

$$\tilde{e}_m(t, x) := e_m(t - t_m, x),$$

so that $[\tilde{e}_{m,t}]$ is also an admissible, eternal forced mean curvature flow of bounded type with forcing term F . By the compactness result of Theorem 1.3.2, there exists an eternal forced mean curvature flow, $[\tilde{e}_{\infty,t}]$, also with forcing term F , towards which $[\tilde{e}_{m,t}]$ subconverges in the C_{loc}^k -sense for all k . However,

$$\mathcal{F}_{\infty}([\tilde{e}_{\infty,0}]) = \lim_{m \rightarrow \infty} \mathcal{F}_m([\tilde{e}_{m,0}]) = \lim_{m \rightarrow \infty} \mathcal{F}([e_{m,t_m}]) = +\infty,$$

which is absurd, and it follows that $\mathcal{F}([e_t])$ is bounded above. In the same manner, $\mathcal{F}([e_t])$ is shown to be bounded below, and this completes the proof. \square

Lemma 2.3.3

Let $[e_t] : \mathbb{R} \rightarrow \mathcal{E}(T^{d+1})$ be an admissible, eternal forced mean curvature flow of bounded type with forcing term F . For any sequence (t_m) converging to $\pm\infty$, there exists an admissible critical point, $[e_\infty]$, of \mathcal{F} towards which the sequence $([e_{t_m}])$ subconverges.

Proof: Observe that the sequence $(\mathcal{F}([e_{t_m}]))$ converges to one of the two boundary points of $I := \mathcal{F}([e_t])$. For all m , define

$$\tilde{e}_m(t, x) := e_m(t - t_m, x),$$

so that $[\tilde{e}_m]$ is also an admissible, eternal forced mean curvature flow of bounded type with forcing term F . By the compactness result of Theorem 1.3.2, there exists an admissible, eternal forced mean curvature flow, $[\tilde{e}_{\infty, t}]$, also with forcing term F , towards which $[\tilde{e}_m]$ subconverges in the C_{loc}^k -sense for all k . However, $\mathcal{F}([\tilde{e}_{\infty, t}]) \subseteq I$, but

$$\mathcal{F}([\tilde{e}_{\infty, 0}]) = \lim_{m \rightarrow \infty} \mathcal{F}([\tilde{e}_{m, 0}]) = \lim_{m \rightarrow \infty} \mathcal{F}([\tilde{e}_{m, t_m}]) \in \partial I,$$

so that $\mathcal{F}([\tilde{e}_{\infty}])$ cannot be open. It follows by Lemma 2.3.2, that $[\tilde{e}_{\infty, t}]$ is a constant flow mapping \mathbb{R} to a critical point of \mathcal{F} . In particular, the sequence, $([e_{t_m}])$, subconverges to $[\tilde{e}_{\infty, 0}]$, and the result follows. \square

The A - and Ω - limit sets of the flow $[e_t]$ are defined by

$$\begin{aligned} A([e_t]) &:= \bigcap_{t \in \mathbb{R}} \overline{\{[e_s] \mid s \leq t\}}, \text{ and} \\ \Omega([e_t]) &:= \bigcap_{t \in \mathbb{R}} \overline{\{[e_s] \mid s \geq t\}}. \end{aligned}$$

By Lemma 2.3.3, these sets are both non-empty and consist only of admissible critical points of \mathcal{F} . Furthermore, being intersections of nested families of connected sets, they are also both connected.

As indicated in the introduction, since admissible critical points of \mathcal{F} are, in particular, constant eternal forced mean curvature flows, the compactness result of Theorem 1.3.2 also applies to these objects. Standard differential-topological arguments then show (c.f., for example, [14]) that, for a generic choice of the density, F , the functional, \mathcal{F} , is of **Morse type** in the sense that all of its admissible critical points are non-degenerate. In particular, in this case, by connectedness, the A - and Ω - limit sets of any admissible, eternal forced mean curvature flow, $[e_t]$, of bounded type and with forcing term F each consist of single points. These points will henceforth be referred to as the **end-points** of $[e_t]$ at plus- and minus- infinity. The results of this section are thus summarised by

Theorem 1.3.3, End-points.

If \mathcal{F} is of Morse type, and if $[e_t]$ is an admissible, forced mean curvature flow of bounded type with forcing term, F , then there exist admissible critical points, $[e_\pm]$, of \mathcal{F} such that

$$\begin{aligned} \lim_{t \rightarrow -\infty} [e_t] &= [e_-], \text{ and} \\ \lim_{t \rightarrow +\infty} [e_t] &= [e_+]. \end{aligned}$$

3 - Admissability.

3.1 - Pinching. Consider an Alexandrov embedding, $e : S^d \rightarrow \mathbb{R}^{d+1}$. Denote its shape operator by A , and let the subscript “;” denote covariant differentiation with respect to the Levi-Civita covariant derivative that it induces over S^d . Recall the generalised Simons’ formula.

Lemma 3.1.1, Generalised Simons’ formula.

For all p, q ,

$$A_{pp;qq} = A_{qq;pp} + A_{pp}^2 A_{qq} - A_{pp} A_{qq}^2.$$

Proof: Indeed, by the Codazzi-Mainardi equations, $A_{ij;k}$ is symmetric under all permutations of the indices. Thus, recalling the definition of curvature,

$$\begin{aligned} A_{ij;kl} &= A_{kj;il} \\ &= A_{kj;li} - R_{lik}^p A_{pj} - R_{lij}^p A_{kp} \\ &= A_{lk;ji} + R_{ilk}^p A_{pj} + R_{ilj}^p A_{kp}, \end{aligned}$$

where R here denotes the Riemann curvature tensor of the metric induced over S^d by the Alexandrov embedding, e . In particular, by Gauss’ equation,

$$R_{ijkl} = A_{il} A_{jk} - A_{ik} A_{jl},$$

so that

$$A_{ij;kl} = A_{lk;ji} + A_{ij}^2 A_{lk} - A_{lk} A_{lj}^2 + A_{ik}^2 A_{lj} - A_{ij} A_{lk}^2,$$

and the result now follows upon substituting $i = j = p$ and $k = l = q$. \square

Let $0 < \kappa_1 \leq \dots \leq \kappa_d$ be the principal curvatures of e . Ideally, the maximum principle would be applied to these functions. However, this would require that they be at least twice differentiable, and since they are, at best, only Lipschitz continuous, they will be smoothed out as follows. Fix a point $(t, x) \in \mathbb{R} \times S^d$, and let ξ_1, \dots, ξ_d be an orthonormal basis of principal directions of e_t at x corresponding to the principal curvatures $\kappa_1, \dots, \kappa_d$ respectively. Extend this basis to an orthonormal frame defined in a neighbourhood of (t, x) which is parallel along S^d at x , and define the smooth functions a_1, \dots, a_d in this neighbourhood by $a_k := \langle A_t \xi_k, \xi_k \rangle$, where A_t is the shape operator of the embedding e_t .

Lemma 3.1.2

At the point (t, x) , for all k ,

$$\left(\partial_t - \frac{1}{d} \Delta \right) a_k = DF(N_t) a_k - a_k^2 F - D^2 F(\xi_k, \xi_k) + \frac{1}{d} a_k \text{Tr}(A^2).$$

Proof: Let $e : \mathbb{R} \times S^d \rightarrow \mathbb{R}^{d+1}$ be a representative of the flow, $[e_t]$, chosen such that

$$\partial_t e_t = (H_t - F \circ e_t) N_t.$$

For fixed vectors, X and Y , tangent to S^d ,

$$\begin{aligned}
 \partial_t(e^*g)(X, Y) &= \partial_t\langle e_*X, e_*Y \rangle \\
 &= \langle \nabla_{\partial_t} e_*X, e_*Y \rangle + \langle e_*X, \nabla_{\partial_t} e_*Y \rangle \\
 &= \langle \nabla_X e_*\partial_t, e_*Y \rangle + \langle e_*X, \nabla_Y e_*\partial_t \rangle \\
 &= 2(H_t - F \circ e_t)\langle A_tX, Y \rangle.
 \end{aligned}$$

Since $e^*g(\xi_k, \xi_k)$ is constant, $\partial_t((e^*g)(\xi_k, \xi_k)) = 0$, and so

$$e^*g(\partial_t\xi_k, \xi_k) = (H_t - F \circ e_t)\langle A_t\xi_k, \xi_k \rangle,$$

and combining these relations yields

$$\partial_t a_k = \langle (\partial_t A)\xi_k, \xi_k \rangle.$$

However, as is well known (c.f. [9]),

$$\partial_t A = -A_t^2(F \circ e_t - H_t) - \text{Hess}(F \circ e_t - H_t),$$

where Hess here denotes the Hessian of functions defined over S^d , so that

$$\partial_t a_k = -a_k^2(F \circ e_t - H_t) - \text{Hess}(F \circ e_t)(\xi_k, \xi_k) + \text{Hess}(H_t)(\xi_k, \xi_k).$$

Furthermore,

$$\text{Hess}(F \circ e_t)(\xi_k, \xi_k) = D^2F(\xi_k, \xi_k) - DF(N_t)a_k,$$

and, by the generalised Simons formula,

$$\begin{aligned}
 \text{Hess}(H_t)(\xi_k, \xi_k) &= \frac{1}{d} \sum_{i=1}^d A_{ii;kk} \\
 &= \frac{1}{d} \sum_{i=1}^d (A_{kk;ii} + A_{ii}^2 A_{kk} - A_{kk}^2 A_{ii}) \\
 &= \frac{1}{d} \Delta a_k + \frac{1}{d} a_k \text{Tr}(A_t^2) - a_k^2 H_t.
 \end{aligned}$$

The result follows upon combining these relations. \square

Lemma 3.1.3

Let $\lambda \geq 1$, $c > 0$ be such that

$$\frac{(\lambda - 1)\lambda}{\lambda + 1} > \frac{\|D^2F\|_{L^\infty}}{c^2 F_-}.$$

If $[e_t]$ is c -convex for all $t \geq 0$, and if $[e_0]$ is pointwise λ -pinched, then $[e_t]$ is pointwise strictly λ -pinched for all $t > 0$.

Proof: Consider the function $\phi : \mathbb{R} \rightarrow \mathbb{R}$ given by

$$\phi(t) := \sup_{x \in S^d} \frac{\kappa_d(t, x)}{\kappa_1(t, x)}.$$

Since this function is Lipschitz continuous, it is almost everywhere differentiable, its derivative is locally L^1 , and it is equal to the integral of its derivative. Let ϕ be differentiable at the point t , and suppose, without loss of generality, that $\phi(t) = \lambda$. Let $x \in S^d$ maximise κ_d/κ_1 and let the functions a_1 and a_d be defined near (t, x) as above. In particular, a_d/a_1 is smooth, $a_d/a_1 \leq \phi$ and $a_d(t, x)/a_1(t, x) = \phi(t)$, so that, at the point (t, x) ,

$$\begin{aligned} \partial_t \log(a_d) - \partial_t \log(a_1) &= \partial_t \log(\phi), \\ \nabla \log(a_d) - \nabla \log(a_1) &= 0, \text{ and} \\ \Delta \log(a_d) - \Delta \log(a_1) &\leq 0. \end{aligned}$$

Thus, by Lemma 3.1.2,

$$\begin{aligned} \partial_t \log(\phi) &= \partial_t \log(a_d) - \partial_t \log(a_1) \\ &\leq (a_1 - a_d)F + \frac{1}{a_1} D^2 F(e_1, e_1) - \frac{1}{a_d} D^2 F(e_d, e_d) \\ &\leq a_1(1 - \lambda)F + \left(1 + \frac{1}{\lambda}\right) \frac{1}{a_1} \|D^2 F\|. \end{aligned}$$

Since $a_1 \geq c$, it follows by hypothesis that this is strictly negative, and this completes the proof. \square

This result also yields a straightforward Hopf-type theorem for eternal forced mean curvature flows.

Theorem 3.1.4, Hopf Theorem for flows.

Let $[e_t] : \mathbb{R} \rightarrow \mathcal{E}(\mathbb{R}^{d+1})$ be an eternal forced mean curvature flow of bounded type with constant forcing term 1. If there exists $\lambda \geq 1$ such that $[e_t]$ is pointwise λ -pinched and λ -non-collapsed for all t , then, up to translation, $[e_t]$, is the constant flow given by

$$e_t(x) = x.$$

Proof: Let λ_0 be the least value in $[1, \infty[$ for which $[e_t]$ is pointwise λ_0 -pinched. We claim that $\lambda_0 = 1$. Indeed, otherwise, via a compactness argument based on Theorem 1.3.2, we may suppose that there exists a point $(t, x) \in \mathbb{R} \times S^d$ such that e_t is λ_0 -pinched at the point x , which is absurd by Lemma 3.1.3. It follows that, for all t , $[e_t]$ is totally umbilic at every point, and is therefore a Euclidean hypersphere. Finally, since the flow, $[e_t]$, is of bounded type, and since it exists for all time, the result now follows by Lemmas 2.1.1 and 2.1.2. \square

3.2 - Non-collapsing. The following result is an adaptation of the non-collapsing argument, [4], of Ben Andrews.

Lemma 3.2.1

Let $[e_t]$ be an eternal forced mean curvature flow with forcing term, f . Let $\lambda \geq 1$, $c > 0$ be such that

$$\frac{(\lambda - 1)\lambda}{(\lambda + 1)} > \frac{\|D^2 F\|_{L^\infty}}{c^2 F_-}.$$

If $[e_t]$ is c -convex and pointwise strictly λ -pinched for all $t \geq 0$, and if $[e_0]$ is λ -non-collapsed, then $[e_t]$ is strictly λ -non-collapsed for all $t > 0$.

Remark: An analogous result also holds for spheres *containing* $[e_0]$. This in turn can be used to provide a more direct proof of a-priori upper bounds of the sort obtained in Lemma 2.2.2. However, since the estimates obtainable by this means are actually weaker, we shall not discuss this further here.

Proof: For a continuous function, $\Phi :]-\epsilon, \epsilon[\times S^d \rightarrow \mathbb{R}$, consider the function, $Z : \mathbb{R} \times S^d \times S^d \rightarrow \mathbb{R}$, given by

$$Z(t, x, y) := \Phi(t, x) \|x - y\|^2 + 2\langle y - x, N_t(x) \rangle,$$

where the points x and y are here identified with their respective images under the embedding e_t . Observe that $[e_t]$ contains the Euclidean hypersphere of curvature $\Phi(t, x)$ which is tangent to $[e_t]$ at the point $e(t, x)$ whenever $Z(t, x, y) \geq 0$ for all $y \in S^d$. Suppose therefore that Z attains a minimum of 0 at the point $y \neq x$, and that Φ is smooth in a neighbourhood of (t, x) . Let R be the reflection orthogonal to $y - x$ which sends x to y . Straightforward geometric arguments yield

$$RN(x) = N(y).$$

Let $\partial x_1, \dots, \partial x_d$ be an orthonormal basis of TS^d at x and let $\partial y_1, \dots, \partial y_d$ be its image under the action of R . That is, for all k ,

$$R\partial x_k = \partial y_k.$$

Observe that $\partial y_1, \dots, \partial y_k$ is also an orthonormal basis of TS^d at y . Extend both (∂x_k) and (∂y_k) to frames defined in neighbourhoods of x and y respectively which are parallel along all geodesics in S^d leaving these points. Consider now the operator

$$\Delta^D := \sum_{k=1}^d (\partial x_k + \partial y_k)^2.$$

The result will follow from the positivity of

$$\left(\partial_t - \frac{1}{d} \Delta^D \right) Z,$$

for a suitable choice of the function Φ . Indeed, first, we have

$$\begin{aligned} D_{\partial x_i + \partial y_i}(y - x) &= \partial y_i - \partial x_i, \\ \frac{1}{d}\Delta^D(y - x) &= H_x N_x - H_y N_y, \\ \partial_t(y - x) &= (F_y - H_y)N_y - (F_x - H_x)N_x, \\ \left(\partial_t - \frac{1}{d}\Delta^D\right)(y - x) &= F_y N_y - F_x N_x. \end{aligned}$$

Likewise,

$$\begin{aligned} D_{\partial x_i + \partial y_i} N_x &= A_i(\partial x_i) \\ \frac{1}{d}\Delta^D N_x &= \nabla^\Sigma H - \frac{1}{d}\text{Tr}(A^2)N_x, \\ \partial_t N_x &= -\nabla^\Sigma(F - H), \\ \left(\frac{\partial}{\partial t} - \frac{1}{d}\Delta^D\right)(y - x) &= -\nabla^\Sigma F + \frac{1}{d}\text{Tr}(A^2)N_x, \end{aligned}$$

so that

$$\begin{aligned} \left(\partial_t - \frac{1}{d}\Delta^D\right) Z &= \left[\left(\partial_t - \frac{1}{d}\Delta^D\right)\Phi\right] \|y - x\|^2 - \frac{4}{d} \sum_{i=1}^d (\partial x_i \Phi) \langle y - x, \partial y_i - \partial x_i \rangle \\ &\quad + 2\Phi \langle F_y N_y - F_x N_x, y - x \rangle - \frac{2}{d} \Phi \sum_{i=1}^d \|\partial y_i - \partial x_i\|^2 \\ &\quad + 2\langle F_y N_y - F_x N_x, N_x \rangle + 2\left\langle y - x, -\nabla^\Sigma F + \frac{1}{d}\text{Tr}(A^2)N_x \right\rangle \\ &\quad - \frac{4}{d} \sum_{i=1}^d \langle \partial y_i - \partial x_i, A_x(\partial x_i) \rangle. \end{aligned}$$

However, since (x, y) minimises Z ,

$$(\partial x_i \Phi) \|y - x\|^2 - 2\Phi \langle y - x, \partial x_i \rangle + 2\langle y - x, \partial x_i \rangle \kappa_i = 0,$$

so that

$$\begin{aligned} \langle \partial x_i, y - x \rangle &= \frac{(\partial x_i \Phi) \|y - x\|^2}{2(\Phi - \kappa_i)}, \\ \langle \partial y_i, y - x \rangle &= \frac{-(\partial x_i \Phi) \|y - x\|^2}{2(\Phi - \kappa_i)}, \\ \langle \partial y_i - \partial x_i, y - x \rangle &= \frac{-(\partial x_i \Phi) \|y - x\|^2}{(\Phi - \kappa_i)}, \\ \|\partial y_i \partial x_i\|^2 &= \frac{(\partial x_i \Phi)^2 \|y - x\|^2}{(\Phi - \kappa_i)^2}, \end{aligned}$$

and combining these relations yields

$$\begin{aligned} \left(\partial_t - \frac{1}{d} \Delta^D \right) Z &= \|y - x\|^2 \left[\left(\partial_t - \frac{1}{d} \Delta \right) \Phi + \frac{2}{d} \sum_{i=1}^d \frac{(\partial x_i \Phi)^2}{(\Phi - \kappa_i)} \right. \\ &\quad \left. + \Phi^2 F - \frac{1}{d} \Phi \text{Tr}(A^2) - \Phi \partial_N F \right] \\ &\quad + 2(F_y - F_x - \langle y - x, \nabla F \rangle) . \end{aligned}$$

In particular, if $\Phi := \lambda a_1$, where a_1 is defined as in Section 3.1, then, by Lemma 3.1.2,

$$\left(\partial_t - \frac{1}{d} \Delta^D \right) Z \geq \|y - x\|^2 (\lambda(\lambda - 1)\kappa_1^2 F - (\lambda + 1)\|D^2 F\|) > 0,$$

as desired. \square

It follows that admissible eternal forced mean curvature flows of bounded type are contained entirely within \mathcal{E}_{λ_F} .

Theorem 1.3.1, Separation.

Let $[e_t] : \mathbb{R} \rightarrow \mathcal{E}(T^{d+1})$ be an eternal forced mean curvature flow of bounded type with forcing term F . If $[e_t]$ is pointwise Λ_F -pinched and Λ_F -non-collapsed for all t , then it is pointwise λ_F -pinched and λ_F -non-collapsed for all t .

Proof: Let λ_0 be the least value in $[0, \Lambda_F]$ for which $[e_t]$ is pointwise λ_0 -pinched and λ_0 -non-collapsed, and suppose that $\lambda_0 \in]\lambda_F, \Lambda_F[$. Via a compactness argument based on Theorem 1.3.2, we may suppose that there exists a point $(t, x) \in \mathbb{R} \times S^d$ such that either e_t is λ_0 -pinched at x or e_t is λ_0 -non-collapsed at x . However, by Lemma 2.1.3, $[e_t]$ is c -convex for all t , where $c := F_-/\lambda_0^2$. Furthermore, since F is subcritical, and since $\lambda_F < \lambda_0 < \Lambda_F$, it follows that the hypotheses of Lemmas 3.1.3 and 3.2.1 are both satisfied. This yields a contradiction, and the result follows. \square

4 - Singular perturbation.

4.1 - Asymptotic series. Recall that the main objective of this text is the proof of Theorem 1.3.5, which describes the asymptotic behaviour of families of admissible, eternal forced mean curvature flows of bounded type as the forcing term tends to infinity in a certain manner. The first step of this proof involves a presentation of the construction of [20], adapted to the case where the ambient manifold is a $(d + 1)$ -dimensional torus. Throughout this and the following section, extensive use will be made of the straightforward fact that the operation of composition by a given smooth function defines smooth maps between Hölder spaces which send bounded sets to bounded sets.

At this stage, various rather technical definitions will be required. First, given a smooth function, $\phi : S^d \rightarrow]-1, \infty[$, denote

$$\hat{\phi}(x) := (1 + \phi(x))x,$$

so that $\hat{\phi}$ is simply the natural parametrisation of the graph of ϕ over S^d . In order to define the functionals of interest to us, it will be useful to introduce the terminology of jet spaces.* For all k , $J^k S^d$ will denote the bundle of k -jets of real-valued functions over S^d . The function $H : J^2 S^d \rightarrow \mathbb{R}$ will be defined such that, for all ϕ , and for all x , $H(J^2 \phi(x))$ will be the mean curvature of the embedding, $\hat{\phi}$, at the point, x . Likewise, the function $N : J^1 S^d \rightarrow S^d$ will be defined such that, for all ϕ , and for all x , $N(J^1 \phi(x))$ will be the outward-pointing, unit, normal vector of the embedding, $\hat{\phi}$, at the point, x . H and N will be both smooth (in fact, analytic) functions defined over the finite-dimensional manifolds, $J^2 S^d$ and $J^1 S^d$, respectively. Furthermore, the Taylor series of H about 0 will be (c.f. [20] and [25])

$$H(J^2 \phi) = 1 - \frac{1}{d}(d + \Delta)\phi + \sum_{m=2}^{\infty} P_m(J_x^2 \phi), \quad (10)$$

where, for all m , P_m is a homogeneous polynomial of order m . Likewise, the Taylor series of N about 0 will be (c.f. [20] and [25])

$$N(J^1 \phi) = x \left(1 + \sum_{m=2}^{\infty} P_{1,m}(J_x^1 \phi) \right) + \left(-\nabla \phi + \sum_{m=2}^{\infty} P_{2,m}(J_x^1 \phi) \right), \quad (11)$$

where, for all m , $P_{1,m}$ and $P_{2,m}$ are homogeneous polynomials of order m .

Now consider a smooth function, $f : T^{d+1} \rightarrow \mathbb{R}$, of Morse-Smale type, and let $\gamma : \mathbb{R} \rightarrow T^{d+1}$ be one of its complete gradient flows (c.f. Section 1.2). For $\kappa > 0$ and for smooth functions, $X : \mathbb{R} \rightarrow \mathbb{R}^{d+1}$ and $\phi : \mathbb{R} \times S^d \rightarrow]-1/\kappa, \infty[$, the smooth function, $\Phi(\kappa, X, \phi) : \mathbb{R} \times S^d \rightarrow T^{d+1}$, will be defined such that

$$\Phi(\kappa, X, \phi)(t, x) := \gamma(t) + \kappa X(t) + \kappa(1 + \kappa^2 \phi(t, x))x, \quad (12)$$

so that, heuristically, $\Phi(\kappa, X, \phi)$ will be a smooth family of embedded spheres each of radius approximately κ , with centres moving along γ , though displaced slightly by the vector field, κX .

For all $\kappa > 0$, the forcing term, $F_\kappa : T^{d+1} \rightarrow \mathbb{R}$, will be defined by

$$F_\kappa := \frac{1}{\kappa}(1 - \kappa^2 f).$$

We now study the conditions to be satisfied by the triplet (κ, X, ϕ) in order for the family $\Phi(\kappa, X, \phi)$ to define an eternal forced mean curvature flow with forcing term, F_κ . We require a further extension of our terminology of jet spaces. Thus, for all k , $J^k(\mathbb{R}, \mathbb{R}^{d+1})$

* Consider a finite-dimensional manifold, M , and given $k > 0$ and a point, $p \in M$, consider the equivalence relation, $\sim_{k,p}$, defined over $C^\infty(M)$ such that $f \sim_{k,p} g$ if and only if they coincide up to order k at the point p . The space $J^k E$ of k -jets of functions over M is defined to be the bundle of all equivalence classes of $\sim_{k,p}$ in $C^\infty(M)$ as p ranges over all points of M . Thus, $J^0 M$ is simply the trivial bundle $\mathbb{R} \times M$, $J^1 M$ is the fibrewise product of $J^0 M$ with $T^* M$, and so on.

will denote the bundle of k -jets of smooth \mathbb{R}^{d+1} -valued functions over \mathbb{R} . For all k , and for any smooth function, $\psi : \mathbb{R} \times S^d \rightarrow \mathbb{R}$, $J_{\text{in}}^k \psi$ will denote the inhomogeneous jet consisting of all derivatives of ψ of order i in x and order j in t for all $i + 2j \leq 2k$. For all k , $J_{\text{in}}^k(\mathbb{R} \times S^d)$ will denote the bundle of inhomogeneous k -jets of real-valued functions over $\mathbb{R} \times S^d$. Observe that $J_{\text{in}}^k(\mathbb{R} \times S^d)$ is also a bundle over \mathbb{R} , and, for all k , J^k will denote its fibrewise product over \mathbb{R} with $J^k(\mathbb{R}, \mathbb{R}^{d+1})$. In other words, a typical element of J^k will have the form $(J^k X(t), J_{\text{in}}^k \phi(t, x))$, for some smooth functions, $X : \mathbb{R} \rightarrow \mathbb{R}^{d+1}$ and $\phi : \mathbb{R} \times S^d \rightarrow \mathbb{R}$, and for some point $(t, x) \in \mathbb{R} \times S^d$.

The function $\Psi :]0, \infty[\times J^1 \rightarrow \mathbb{R}$ will be defined such that, for all $\kappa \in]0, \infty[$, for all smooth functions, $X : \mathbb{R} \rightarrow \mathbb{R}^{d+1}$ and $\phi : \mathbb{R} \times S^d \rightarrow \mathbb{R}$, and for any point $(t, x) \in \mathbb{R} \times S^d$,

$$\begin{aligned} \Psi(\kappa, J^1 X(t), J_{\text{in}}^1 \phi(t, x)) &:= \kappa^2 \langle D_t \Phi(\kappa, X, \phi)(t, x), N(\kappa^2 J_x^1 \phi(t, x)) \rangle + \\ &\quad \frac{1}{\kappa} H(\kappa^2 J_x^2 \phi(t, x)) - F_\kappa(\Phi(\kappa, X, \phi)(t, x)), \end{aligned} \quad (13)$$

where $J_x^1 \phi$ and $J_x^2 \phi$ here denote respectively the first and second order jets of derivatives of ϕ in the x direction. As with H and N , Ψ will be a smooth function defined over the finite-dimensional manifold, $]0, \infty[\times J^1$. By (5), we see that this function has been constructed precisely in order that, given $\kappa > 0$, and smooth functions, $X : \mathbb{R} \rightarrow \mathbb{R}^{d+1}$ and $\phi : \mathbb{R} \times S^d \rightarrow \mathbb{R}$, the family $\Phi(\kappa, X, \phi)$ is an eternal forced mean curvature flow with forcing term, F_κ , if and only if $\Psi(\kappa, J^1 X(t), J_{\text{in}}^1 \phi(t, x))$ vanishes for all $(t, x) \in \mathbb{R} \times S^d$.

Solutions of the forced mean curvature flow equation will now be obtained in the standard manner by first constructing formal solutions which are then perturbed to exact solutions via the inverse function theorem. To this end, the following terminology of asymptotic series will also prove useful. Consider an arbitrary, finite-dimensional vector bundle, E , over some base, B . Given a smooth function, $\phi :]0, \infty[\times E \rightarrow \mathbb{R}$, and a sequence, (ϕ_i) , of smooth functions where, for all i , $\phi_i : E^{\otimes i} \rightarrow \mathbb{R}$, and a formal power series, $\xi(\kappa, x) \sim \sum_{i=0}^{\infty} \kappa^i \xi_i(x)$, taking values in E , the expression

$$\phi(\kappa, \xi) \sim \sum_{i=0}^{\infty} \kappa^i \phi_i(\xi_{0,x}, \dots, \xi_{i,x}) \quad (14)$$

will mean that, for all $m \geq 0$, there exists a smooth function, $R_m : [0, \infty[\times E^{\otimes m} \rightarrow \mathbb{R}$, such that

$$\phi\left(\kappa, \sum_{i=0}^m \kappa^i \xi_i(x)\right) = \sum_{i=0}^m \kappa^i \phi_i(\xi_0(x), \dots, \xi_i(x)) + \kappa^{m+1} R_m(\kappa, \xi_0(x), \dots, \xi_m(x)).$$

The series $\sum_{i=0}^{\infty} \kappa^i \phi_i$ will be called the **asymptotic series** of the function, ϕ . It will be of fundamental importance here that the remainder term, R_m , be smooth at the point $t = 0$. Indeed, otherwise, this relation would be trivially satisfied for any pair of sequences, (ϕ_i) and (ξ_i) , making it of little use to us.

Lemma 4.1.1

There exists a sequence, (Ψ_i) , of functions, which are polynomials in all but the first variable, such that, for all formal series, $X \sim \sum_{i=0}^{\infty} \kappa^i X_i$ and $\phi \sim \sum_{i=0}^{\infty} \kappa^i \phi_i$,

$$\begin{aligned}
 \Psi(\kappa, X, \phi) \sim & -\frac{1}{2}\kappa^3 \text{Hess}(f)(\gamma(t))(X_0, X_0) - \frac{1}{2}\kappa^3 \text{Hess}(f)(\gamma(t))(x, x) \\
 & + \kappa \sum_{i=0}^{\infty} \kappa^i \left[\langle D_t X_{i-2} - \text{Hess}(f)(\gamma(t))X_{i-2}, x \rangle \right. \\
 & \quad + \left(\kappa^4 D_t - \frac{1}{d}(d + \Delta) \right) \phi_i \\
 & \quad + \langle \nabla f(\gamma(t)), X_{i-1} \rangle + \text{Hess}(f)(\gamma(t))(X_{i-2}, X_0) \\
 & \quad + \Psi_m(\gamma, D_t \gamma, x, J_t^1 X_0, \dots, J_t^1 X_{i-3}, \\
 & \quad \left. \kappa^4 D_t \phi_0, \dots, \kappa^4 D_t \phi_{i-2}, J_x^2 \phi_0, \dots, J_x^2 \phi_{i-2} \right),
 \end{aligned} \tag{15}$$

where, by convention, X_i and ϕ_i are taken to be equal to 0 for i negative. Furthermore,

$$\begin{aligned}
 \Psi_0 &= f(\gamma(t)), \text{ and} \\
 \Psi_1 &= 0.
 \end{aligned}$$

Proof: By (10), there exists a sequence, (H_i) , of polynomials such that

$$H(\kappa^2 J_x^2 \phi) \sim 1 - \kappa^2 \sum_{i=0}^{\infty} \frac{1}{d}(d + \Delta) \kappa^i \phi_i + \kappa^4 \sum_{i=0}^{\infty} \kappa^i H_i(J_x^2 \phi_0, \dots, J_x^2 \phi_i).$$

Likewise, by (11), there exist sequences, $(N_{1,i})$ and $(N_{2,i})$, of polynomials such that

$$N(x, \kappa^2 J_x^1 \psi) = x(1 + N_1(J_x^2 \psi)) + \kappa^2 N_2(J_x^1 \psi),$$

where

$$\begin{aligned}
 N_1(\kappa^2 J_x^1 \psi) &\sim \kappa^4 \sum_{i=0}^{\infty} \kappa^i N_{1,i}(J_x^1 \psi_0, \dots, J_x^1 \psi_i), \text{ and} \\
 N_2(\kappa^2 J_x^1 \psi) &\sim - \sum_{i=0}^{\infty} \kappa^i \nabla \psi_i + \kappa^2 \sum_{i=0}^{\infty} \kappa^i N_{2,i}(J_x^1 \psi_0, \dots, J_x^1 \psi_i).
 \end{aligned}$$

Finally, by Taylor's theorem, there exists a sequence, (F_i) , of functions, which are polyno-

mials in all but the first variable, such that

$$\begin{aligned}
 F_\kappa(\Phi(\kappa, X, \phi)) &\sim \frac{1}{\kappa} - \kappa f(\gamma(t)) - \kappa^2 \langle \nabla f(\gamma(t)), x \rangle \\
 &\quad - \kappa^2 \sum_{i=0}^{\infty} \kappa^i \langle \nabla f(\gamma(t)), X_i \rangle \\
 &\quad - \frac{1}{2} \kappa^3 \langle \text{Hess}(f)(\gamma(t))x, x \rangle - \frac{1}{2} \kappa^3 \sum_{i,j=0}^{\infty} \kappa^{i+j} \langle \text{Hess}(f)(\gamma(t))X_i, X_j \rangle \\
 &\quad - \kappa^3 \sum_{i=0}^{\infty} \kappa^i \langle \text{Hess}(f)(\gamma(t))X_i, x \rangle \\
 &\quad + \kappa^4 \sum_{i=0}^{\infty} F_i(\gamma(t), x, J_t^0 X_0, \dots, J_t^0 X_i, J_x^0 \phi_0, \dots, J_x^0 \phi_i).
 \end{aligned}$$

The result now follows upon combining these relations with the definitions (12) and (13) of Φ and Ψ respectively. \square

4.2 - Spherical harmonics and formal solutions. Consider the standard Laplace-Beltrami operator, Δ , of functions over S^d . For all integer $m \geq 0$, the vector space, $\mathcal{H}_m := \mathcal{H}_m(S^d)$, of eigenfunctions of Δ with eigenvalue $m(m+d-1)$ will be referred to as the space of m 'th order, d -dimensional **spherical harmonics**. Since every eigenfunction of Δ is a spherical harmonic of some order, the spherical harmonics together form an orthonormal basis of $L^2(S^d)$. For all m , every m 'th order spherical harmonic is the restriction to S^d of a unique homogeneous, harmonic polynomial of degree m . Conversely, the restriction to S^d of any polynomial of degree m decomposes uniquely as a sum of spherical harmonics of orders at most m . In addition, if this polynomial is even (resp. odd), then every non-trivial component of this decomposition has even (resp. odd) order.

The mean curvature flow equation, $\Psi(\kappa, J^1 X(t), J_{\text{in}}^1 \phi(t, x)) = 0$, introduced in the previous section, exhibits two distinct asymptotic behaviours as the parameter, κ , tends to 0. This will be resolved by rescaling the space, \mathcal{H}_1 , of first-order spherical harmonics, as the parameter, κ , tends to 0 (c.f. Section 4.4, below). To this end, we will denote

$$\hat{\mathcal{H}} := \oplus_{m \neq 1} \mathcal{H}_m,$$

and $\Pi : L^2(S^d) \rightarrow \mathcal{H}_1$ and $\Pi^\perp : L^2(S^d) \rightarrow \hat{\mathcal{H}}$ will denote the orthogonal projections. In like manner, for all k , we will denote

$$\hat{\mathcal{H}}_k := \oplus_{m \neq 1, m \leq k} \mathcal{H}_m.$$

In addition, $C_{\text{bdd}}^\infty(\mathbb{R} \times S^d)$ will denote the space of smooth functions, $f : \mathbb{R} \times S^d \rightarrow \mathbb{R}$, whose derivatives to all orders are bounded, and $\hat{C}_{\text{bdd}}^\infty(\mathbb{R} \times S^d)$ will denote the subspace consisting of those functions, $f : \mathbb{R} \times S^d \rightarrow \mathbb{R}$, such that, for all t ,

$$\Pi(f(t, \cdot)) = 0.$$

Similarly, for any finite-dimensional vector space, E , $C_{\text{bdd}}^\infty(\mathbb{R}, E)$ will denote the space of smooth functions, $f : \mathbb{R} \rightarrow E$, whose derivatives to all orders are bounded. In particular, for all $k \neq 1$, $C_{\text{bdd}}^\infty(\mathbb{R}, \hat{\mathcal{H}}_k)$ canonically identifies with a subspace of $\hat{C}_{\text{bdd}}^\infty(\mathbb{R} \times S^d)$.

Consider now the operators that arise in the asymptotic series, (15), of Ψ . First, the operator, $L : C_{\text{bdd}}^\infty(\mathbb{R}, \mathbb{R}^{d+1}) \rightarrow C_{\text{bdd}}^\infty(\mathbb{R}, \mathbb{R}^{d+1})$, is defined by

$$L := D_t + \text{Hess}(f)(\gamma(t)). \quad (16)$$

Since f is of Morse type, this operator is of Fredholm type with Fredholm index equal to the difference of the Morse indices of its end-points (c.f. [17]). Furthermore, elements of its kernel decay exponentially at $\pm\infty$, so that the L^2 -orthogonal complement of this kernel in $C_{\text{bdd}}^\infty(\mathbb{R}, \mathbb{R}^{d+1})$ is well-defined. Finally, since f is of Morse-Smale type, this operator is surjective, and therefore has a unique left-inverse, K , which sends $C_{\text{bdd}}^\infty(\mathbb{R}, \mathbb{R}^{d+1})$ into $\text{Ker}(L)^\perp$.

Next, for all $\kappa > 0$, the operator, $P_\kappa : C_{\text{bdd}}^\infty(\mathbb{R} \times S^d) \rightarrow C_{\text{bdd}}^\infty(\mathbb{R} \times S^d)$, is defined by

$$P_\kappa := \kappa^4 D_t - \frac{1}{d} (d + \Delta). \quad (17)$$

It follows from the classical theory of parabolic operators that P_κ defines an invertible map from $\hat{C}_{\text{bdd}}^\infty(\mathbb{R} \times S^d)$ to itself. Let Q_κ denote its inverse. Observe, furthermore, that, for all $k \neq 1$, both P_κ and Q_κ restrict to linear isomorphisms of $C_{\text{bdd}}^\infty(\mathbb{R}, \hat{\mathcal{H}}_k)$.

Formal solutions of the forced mean curvature flow equation are now obtained in a straightforward manner. Oddly, the coefficients of the asymptotic series of X and ϕ actually depend on κ , although, as we will show presently, they are nonetheless uniformly bounded in every norm.

Lemma 4.2.1

There exist unique formal series, $X \sim \sum_{i=0}^\infty \kappa^i X_{i,\kappa}$ in $\text{Ker}(L)^\perp$, and $\phi \sim \sum_{i=0}^\infty \kappa^i \phi_{i,\kappa}$ in $\hat{C}_{\text{bdd}}^\infty(\mathbb{R} \times S^d)$, formally solving

$$\Phi(\kappa, J^1 X(x), J_{\text{in}}^1 \phi(t, x)) \sim 0.$$

Furthermore, for all i , there exists k_i such that ϕ_i is an element of $C_{\text{bdd}}^\infty(\mathbb{R}, \hat{\mathcal{H}}_{k_i})$.

Proof: Suppose, by induction, that $X_{0,\kappa}, \dots, X_{i,\kappa}$ and $\phi_{0,\kappa}, \dots, \phi_{i,\kappa}$ have been constructed. In particular, the orthogonal projections onto $\hat{\mathcal{H}}$ of terms in the series (15) of order up to and including $(i+1)$ in κ all vanish. Likewise, the orthogonal projections onto \mathcal{H}_1 of terms in this series of order up to and including $(i+3)$ in κ also all vanish. Define

$$\begin{aligned} \phi_{i+1,\kappa} := & -Q_\kappa(\langle \nabla f(p), X_{i,\kappa} \rangle + \text{Hess}(f)(p)(X_{i-1,\kappa}, X_{0,\kappa}) \\ & + \Pi_1^\perp \Psi_i(D_t \gamma, x, J_t^1 X_{0,\kappa}, \dots, J_t^1 X_{i-2,\kappa}, \\ & \kappa^4 D_t \phi_{0,\kappa}, \dots, \kappa^4 D_t \phi_{i-1,\kappa}, J_x^2 \phi_{0,\kappa}, \dots, J_x^2 \phi_{i-1,\kappa})) . \end{aligned} \quad (18)$$

In particular, since Ψ_i is a polynomial in all but the first variable, it follows by the inductive hypothesis that $\phi_{i+1,\kappa}$ is an element of $C_{\text{bdd}}^\infty(\mathbb{R}, \hat{\mathcal{H}}_{k_{i+1}})$, for some k_{i+1} independent of κ . Define

$$X_{i+1,\kappa} := -L \left(\Pi_1 \Psi_i(D_r \gamma, x, J_t^1 X_{0,\kappa}, \dots, J_t^1 X_{i,\kappa}, \kappa^4 D_t \phi_{0,\kappa}, \dots, \kappa^4 D_t \phi_{i+1,\kappa}, J_x^2 \phi_{0,\kappa}, \dots, J_x^2 \phi_{i+1,\kappa}) \right). \quad (19)$$

With $\phi_{i+1,\kappa}$ defined as above, the orthogonal projection onto $\hat{\mathcal{H}}$ of the term in the series (15) of order $(i+2)$ in κ vanishes. Likewise, with $X_{i+1,\kappa}$ defined as above, the orthogonal projection onto \mathcal{H}_1 of the term in this series of order $(i+4)$ in κ also vanishes. Furthermore, by definition of Q_κ and L , $\phi_{i+1,\kappa}$ and $X_{i+1,\kappa}$ are the only functions with these properties, and the result now follows. \square

4.3 - The κ -dependent rescaled time variable. Consider a finite-dimensional vector space, E , and a linear operator, $M : C_{\text{bdd}}^1(\mathbb{R}, E) \rightarrow C_{\text{bdd}}^0(\mathbb{R}, E)$, of the form

$$Mf = D_t f + A(t)f,$$

where $A : \mathbb{R} \rightarrow \text{End}(E)$ is a smooth function whose derivatives to all orders are bounded, and such that

$$\lim_{t \rightarrow \pm\infty} A(t) = A_\pm,$$

for some invertible operators A_\pm . These hypotheses imply that M is of Fredholm type (c.f. [17]). Furthermore, all derivatives of all elements of its kernel decay exponentially at plus- and minus- infinity, and, in particular, the L^2 -orthogonal projection, Π , from $C_{\text{bdd}}^0(\mathbb{R}, E)$ onto this kernel is well-defined. Let $\|\cdot\|_0$ denote the uniform norm over $C_{\text{bdd}}^0(\mathbb{R}, E)$. Straightforward integral estimates yield

Lemma 4.3.1

There exists a constant, $B > 0$, such that, for all $f \in C_{\text{bdd}}^1(\mathbb{R}, E)$,

$$\|f\|_0, \|D_t f\|_0 \leq B (\|Mf\|_0 + \|\Pi f\|_0).$$

As can be seen from (16) and (17), different components of the flow, $\Phi(\kappa, X, \phi)$, evolve at different rates, depending on the parameter, κ . For this reason, it will be useful to introduce the **κ -dependent, rescaled time variable**, $s := \kappa^{-4}t$, so that the operator of differentiation with respect to s will then be given by $D_s := \kappa^4 D_t$.

For all $\alpha \in [0, 1]$, the κ -dependent **Hölder difference operator** of order α will be given by

$$\delta_s^\alpha f(t) := \sup_{|h| \leq 1} \frac{|f(t + \kappa^4 h) - f(t)|}{h^\alpha}.$$

Likewise, for all (k, α) , the κ -dependent **Hölder norm** of order (k, α) will be given by

$$\|f\|_{k,\alpha,\kappa} := \sum_{i=0}^k \|D_s^i f\|_0 + \|\delta_s^\alpha D_s^k f\|_0.$$

Finally, the **Hölder space** of order (k, α) will be defined by

$$C^{k,\alpha}(\mathbb{R}, E) := \{f \in C^k(\mathbb{R}, E) \mid \|f\|_{k,\alpha,\kappa} < \infty\}.$$

Observe that, for any given (k, α) , the κ -dependent Hölder norms of order (k, α) are all pairwise uniformly equivalent, so that the space, $C^{k,\alpha}(\mathbb{R}, E)$, defined as above, is indeed independent of κ .

An important fact to be used in the sequel is that the norm of the Green's operator of M is actually uniformly bounded, independent of the parameter, κ . Indeed,

Lemma 4.3.2

For all (k, α) , there exists $B > 0$ such that, for all $\kappa \in]0, 1]$, and for all $f \in C^{k+1,\alpha}(\mathbb{R}, E)$,

$$\|f\|_{k,\alpha,\kappa}, \|D_t f\|_{k,\alpha,\kappa} \leq B (\|Mf\|_{k,\alpha,\kappa} + \|\Pi f\|_0).$$

Proof: Observe that, for all (k, α) , there exists $B_{k,\alpha} > 0$ such that, for all $\kappa \in]0, 1]$,

$$\|\delta_s^\alpha D_s^k A\|_0 \leq B_{k,\alpha} \kappa^{4(k+\alpha)} \leq B_{k,\alpha}. \quad (20)$$

Let h_1, \dots, h_m be an L^2 -orthonormal basis of $\text{Ker}(M)$. Recall (c.f. [17]), that, for all i , and for all k , $D_t^k h_i$ decays exponentially at plus- and minus- infinity, so that, upon increasing $B_{k,\alpha}$ if necessary, we may suppose that,

$$\int_{\mathbb{R}} \|\delta_s^\alpha D_s^k h_i\| dt \leq B_{k,\alpha} \kappa^{4(k+\alpha)} \leq B_{k,\alpha}. \quad (21)$$

Now suppose, by induction, that the result holds for all $0 \leq k \leq l$. Then,

$$\|D_s^{l+1} f\|_0 \leq \|MD_s f\|_{l,\alpha,\kappa} + \|\Pi D_s f\|_0.$$

However,

$$MD_s f = D_s M f + (D_s A) f,$$

so that, by (20), and the inductive hypothesis,

$$\begin{aligned} \|MD_s f\|_{l,\alpha,\kappa} &\leq B_1 (\|D_s M f\|_{l,\alpha,\kappa} + \|f\|_{l,\alpha,\kappa}) \\ &\leq B_2 (\|Mf\|_{l+1,\alpha,\kappa} + \|\Pi(f)\|_0), \end{aligned}$$

for some constants, $B_1, B_2 > 0$. Likewise, by (21), and the inductive hypothesis again, for all i ,

$$\begin{aligned} \left| \int_{\mathbb{R}} (D_s f) h_i dt \right| &= \left| \int_{\mathbb{R}} f (D_s h_i) dt \right| \\ &\leq B_3 \|f\|_0 \\ &\leq B_4 (\|Mf\|_{l+1,\alpha,\kappa} + \|\Pi(f)\|_0), \end{aligned}$$

for some constants, $B_3, B_4 > 0$. Upon combining these relations, it follows that

$$\|D_s^{l+1}f\|_0 \leq B_5 (\|Mf\|_{l+1,\alpha,\kappa} + \|\Pi f\|_0),$$

for some constant, $B_5 > 0$. Furthermore, using (20) and the inductive hypothesis once again,

$$\begin{aligned} \|D_t D_s^{l+1}f\|_0 &= \|D_s^{l+1}D_t f\|_0 \\ &\leq \|D_s^{l+1}Mf\|_0 + \|D_s^{l+1}Af\|_0 \\ &\leq B_6 (\|Mf\|_{l+1,\alpha,\kappa} + \|\Pi f\|_0), \end{aligned}$$

for some constant, $B_6 > 0$. The estimates for $\delta_s^\alpha D_s^{l+1}f$ and $\delta_s^\alpha D_s^{l+1}D_t f$ are obtained in a similar manner, and the result now follows by induction. \square

It will also be useful to introduce κ -dependent Hölder norms for spaces of functions defined over $\mathbb{R} \times S^d$. Thus, given $\alpha \in]0, 1]$, the κ -dependent **Hölder difference operators** of order α are given by

$$\begin{aligned} \delta_x^\alpha f(t, x) &:= \sup_{y \neq x} \frac{|f(t, y) - f(t, x)|}{\|x - y\|^\alpha}, \text{ and} \\ \delta_s^\alpha f(t, x) &:= \sup_{|h| \leq 1} \frac{|f(t + \kappa^4 h, x) - f(t, x)|}{h^\alpha}. \end{aligned}$$

For all $k \in \mathbb{N}$, $C_{\text{in}}^k(\mathbb{R} \times S^d)$ will denote the space of all functions, $f : \mathbb{R} \times S^d \rightarrow \mathbb{R}$ which are continuously differentiable i times in the x direction and j times in the t direction for all $i + 2j \leq 2k$. For all $k \in \mathbb{N}$ and for all $\alpha \in]0, 1/2]$, the κ -dependent **inhomogeneous Hölder norm** of order (k, α) over $C_{\text{in}}^k(\mathbb{R} \times S^m)$ will be defined by

$$\|f\|_{k,\alpha,\kappa,\text{in}} := \sum_{i+2j \leq 2k} \|D_x^i D_s^j f\|_0 + \sum_{i+2j=2k} \|\delta_x^{2\alpha} D_x^i D_s^j f\|_0 + \sum_{i+2j=2k} \|\delta_s^\alpha D_x^i D_s^j f\|_0.$$

For all (k, α) , the **inhomogeneous Hölder space** of order (k, α) will be defined by

$$C_{\text{in}}^{k,\alpha}(\mathbb{R} \times S^d) := \{f \in C_{\text{in}}^k(\mathbb{R} \times S^d) \mid \|f\|_{k,\alpha,\kappa,\text{in}} < \infty\}.$$

As before, for any given (k, α) , the κ -dependent inhomogeneous Hölder norms of order (k, α) are all pairwise uniformly equivalent, so that the space, $C_{\text{in}}^{k,\alpha}(\mathbb{R} \times S^d)$, as defined above, is indeed independent of κ .

Uniform estimates for the formal solutions constructed in the preceeding section now follow.

Lemma 4.3.3

For all (k, α) , and for all i , there exists $B_{k,\alpha,i} > 0$ such that, for all $\kappa \in]0, 1]$,

$$\begin{aligned} \|X_{i,\kappa}\|_{k,\alpha,\kappa} &\leq B_{k,\alpha,i}, \text{ and} \\ \|\phi_{i,\kappa}\|_{k,\alpha,\kappa,\text{in}} &\leq B_{k,\alpha,i}. \end{aligned}$$

Proof: Indeed, suppose, by induction, that the result holds for all $i \leq j$. Recall that composition by a given smooth function defines smooth maps between Hölder spaces which send bounded sets to bounded sets. Thus, since Ψ_j is itself a smooth function defined over some finite-dimensional manifold, it follows by (18), (19) and the inductive hypothesis that

$$\begin{aligned} \|P_\kappa \phi_{j+1, \kappa}\|_{k, \alpha, \kappa, \text{in}} &\leq B_{k, \alpha, j+1}, \text{ and} \\ \|LX_{j+1, \kappa}\|_{k, \alpha, \kappa} &\leq B_{k, \alpha, j+1}, \end{aligned}$$

for some constant, $B_{k, \alpha, j+1} > 0$. The estimate for $X_{j+1, \kappa}$ now follows by Lemma 4.3.2. However, for all κ , $P_\kappa = D_s - \frac{1}{d}(d + \Delta)$, so that the norm of the operator, Q_κ , with respect to the κ -dependent Hölder norms, $\|\cdot\|_{k, \alpha, \kappa, \text{in}}$ and $\|\cdot\|_{k+1, \alpha, \kappa, \text{in}}$, is actually independent of κ . The estimate for $\phi_{j+1, \kappa}$ therefore follows, and this completes the proof. \square

4.4 - Exact solutions. Consider now the map $\hat{\Psi}_\kappa : C^{k+1, \alpha}(\mathbb{R}, \mathbb{R}^{d+1}) \times \hat{C}_{\text{in}}^{k+1, \alpha}(\mathbb{R} \times S^d) \rightarrow C_{\text{in}}^{k, \alpha}(\mathbb{R} \times S^d)$ given by

$$\hat{\Psi}_\kappa(X, \phi)(t, x) := \frac{1}{\kappa} \Psi(\kappa, J^1 X(t), J_{\text{in}}^1 \phi(t, x)).$$

Since $\hat{\Psi}_\kappa$ is constructed via a combination of differentiation and composition by smooth functions, it defines a smooth map between Hölder spaces. Recall that, as outlined in Section 4.2, for all (k, α) , the space $C_{\text{in}}^{k, \alpha}(\mathbb{R} \times S^d)$ naturally decomposes as the direct sum

$$C_{\text{in}}^{k, \alpha}(\mathbb{R} \times S^d) = C^{k, \alpha}(\mathbb{R}, \mathbb{R}^{d+1}) \oplus \hat{C}_{\text{in}}^{k, \alpha}(\mathbb{R} \times S^d).$$

With respect to this decomposition, the linear isomorphism $T_\kappa : C_{\text{in}}^{k, \alpha}(\mathbb{R} \times S^d) \rightarrow C_{\text{in}}^{k, \alpha}(\mathbb{R} \times S^d)$ is defined by

$$T_\kappa := \begin{pmatrix} \kappa^{-2} \text{Id} & \\ & \text{Id} \end{pmatrix}.$$

Using this operator, uniform a-priori bounds for the inverse of the derivative of $\hat{\Psi}$ are obtained.

Lemma 4.4.1

For all (k, α) , and for all $R > 0$, there exists $B > 0$ such that, for sufficiently small κ , and for all $\|X\|_{k+1, \alpha, \kappa} + \|\phi\|_{k+1, \alpha, \kappa, \text{in}} < R$, the operator, $T_\kappa D\hat{\Psi}_\kappa(X, \phi)$, defines a linear isomorphism from $C^{k+1, \alpha}(\mathbb{R}, \mathbb{R}^{d+1}) \times \hat{C}_{\text{in}}^{k+1, \alpha}(\mathbb{R} \times S^d)$ into $C_{\text{in}}^{k, \alpha}(\mathbb{R} \times S^d)$, such that

$$\left\| \left[T_\kappa D\hat{\Psi}_\kappa(X, \phi) \right]^{-1} \right\| \leq B.$$

Proof: Indeed, consider first the separate components of the derivative of $\hat{\Psi}_\kappa$. By (10), the derivative of $\kappa^{-2}H(\kappa^2 J_x^2 \phi)$ is given by

$$\kappa^{-2}DH(\kappa^2 J_x^2 \phi)\psi = -\frac{1}{d}(d + \Delta)\psi + \kappa^2 N_1 J_x^2 \psi,$$

where N_1 is a multiplication operator of norm bounded in terms of R . By (11), the derivative of $N(\kappa^2 J_x^1 \phi)$ is given by

$$DN(\kappa^2 J_x^1 \phi)\psi = \kappa^2 N_2 J_x^1 \psi,$$

where N_2 is a multiplication operator of norm bounded in terms of R . Thus

$$\begin{aligned} \kappa D\langle D_t \Phi_\kappa(X, \phi), N(\kappa^2 J_x^1 \phi) \rangle(Y, \psi) &= \kappa^2 \langle D_t Y, x \rangle + \kappa^4 D^t \psi \\ &\quad + \kappa^2 N_3(D_s \psi) + \kappa^4 N_4 J_x^1 \psi + \kappa^4 N_5 D_t Y, \end{aligned}$$

where N_3 , N_4 and N_5 are multiplication operators of norms bounded in terms of R . Finally, the derivative of $\frac{1}{\kappa} F(\Phi_\kappa(X, \phi))$ is given by

$$\begin{aligned} \frac{1}{\kappa} DF(\Phi_\kappa(X, \phi)) &= -\kappa \langle \nabla f(\gamma(t)), Y \rangle - \kappa^2 \langle \text{Hess}(f)(\gamma(t))Y, x \rangle \\ &\quad - \kappa^2 \langle \text{Hess}(\gamma(t))Y, X \rangle + \kappa^3 N_6 Y + \kappa^3 N_7 \psi, \end{aligned}$$

where N_6 and N_7 are multiplication operators of norms bounded in terms of R . Combining these relations, it follows that, with respect to the above decomposition of $C_{\text{in}}^{k, \alpha}(\mathbb{R} \times S^d)$, the derivative, $D\hat{\Psi}_\kappa(X, \phi)$, satisfies

$$T_\kappa D\hat{\Psi}_\kappa(X, \phi) = \begin{pmatrix} L & M_1 \\ & P_\kappa \end{pmatrix} + \begin{pmatrix} \kappa M_2 & \kappa M_3 \\ \kappa M_4 & \kappa^2 M_5 \end{pmatrix},$$

where, for all $1 \leq i \leq 5$, the linear operator, M_i , satisfies

$$\|M_i\| \leq B,$$

for some constant, $B > 0$, which only depends on R . The result now follows by Lemma 4.3.2 and the fact that the norm of Q_κ with respect to the κ -dependent Hölder norms, $\|\cdot\|_{k, \alpha, \kappa, \text{in}}$ and $\|\cdot\|_{k+1, \alpha, \kappa, \text{in}}$, is independent of κ . \square

Consider now a Banach space, E . Given a formal power series, $\sum_{i=0}^{\infty} \kappa^i f_{i, \kappa}$, taking values in E , and a function, $f :]0, \epsilon[\rightarrow E$, the expression

$$f \sim \sum_{i=0}^{\infty} \kappa^i f_{i, \kappa}$$

will mean that for all $m \geq 0$, there exists a constant, B_m , such that, for all $\kappa \in]0, 1]$,

$$\left\| f - \sum_{i=0}^m \kappa^i f_{i, \kappa} \right\| \leq B_m \kappa^{m+1}.$$

The series, $\sum_{i=0}^{\infty} \kappa^i f_{i, \kappa}$, will be called the **norm-asymptotic** series of the function, f , whenever this holds.

Theorem 4.4.2

For sufficiently small $\epsilon > 0$, there exist functions $X :]0, \epsilon[\rightarrow C^{k+1, \alpha}(\mathbb{R}, \mathbb{R}^{d+1})$ and $\phi :]0, \epsilon[\rightarrow \hat{C}_{\text{in}}^{k+1, \alpha}(\mathbb{R} \times S^d)$ such that

$$\begin{aligned} X &\sim \sum_{i=0}^{\infty} \kappa^i X_{i, \kappa}, \\ \phi &\sim \sum_{i=0}^{\infty} \kappa^i \phi_{i, \kappa}. \end{aligned}$$

and

$$\Psi(\kappa, X_{\kappa}, \phi_{\kappa}) = 0.$$

Furthermore, X and ϕ are unique in the sense that, if $X' :]0, \epsilon[\rightarrow C^{k+1, \alpha}(\mathbb{R}, \mathbb{R}^{d+1})$ and $\phi' :]0, 1[\rightarrow C_{\text{in}}^{k+1, \alpha}(\mathbb{R} \times S^d)$ are other functions with the same properties then, for sufficiently small κ , $X(\kappa) = X'(\kappa)$, and $\phi(\kappa) = \phi'(\kappa)$.

Remark: In particular, for sufficiently small κ , the eternal forced mean curvature flows constructed in Theorem 4.4.2 are all admissible, of bounded type, and even non-degenerate, in the sense that their linearised mean curvature flow operators are surjective.

Proof: Let m be a positive integer and define

$$\begin{aligned} \tilde{X}_{m, \kappa} &:= \sum_{i=0}^m \kappa^i X_{i, \kappa}, \text{ and} \\ \tilde{\phi}_{m, \kappa} &:= \sum_{i=0}^m \kappa^i \phi_{i, \kappa}. \end{aligned}$$

Consider the asymptotic formula, (15), for Ψ , substituting $X_i = X_{i, \kappa}$ and $\phi_i = \phi_{i, \kappa}$ for $0 \leq i \leq m$, and $X_i = 0$ and $\phi_i = 0$ for $i > m$. By definition of the formal series $\sum_{i=0}^{\infty} \kappa^i X_{i, \kappa}$ and $\sum_{i=0}^{\infty} \kappa^i \phi_{i, \kappa}$, there exist smooth functions $R_{1, m}$ and $R_{2, m}$ such that

$$\begin{aligned} \Pi \hat{\Psi}_{\kappa}(\tilde{X}_{m, \kappa}, \tilde{\phi}_{m, \kappa}) &= \kappa^{m+3} \Pi(R_{1, m}(\gamma, D_t \gamma, x, J^1 X_{0, \kappa}(t), \dots, J^1 X_{m, \kappa}(t), \\ &\quad \kappa^4 D_t \phi_{0, \kappa}(t, x), \dots, \kappa^4 D_t \phi_{m, \kappa}(t, x), \\ &\quad J_x^2 \phi_{0, \kappa}(t, x), \dots, J_x^2 \phi_{m, \kappa}(t, x))), \text{ and} \\ \Pi^{\perp} \hat{\Psi}_{\kappa}(\tilde{X}_{m, \kappa}, \tilde{\phi}_{m, \kappa}) &= \kappa^{m+1} \Pi^{\perp}(R_{2, m}(\gamma, D_t \gamma, x, J^1 X_{0, \kappa}(t), \dots, J^1 X_{m, \kappa}(t), \\ &\quad \kappa^4 D_t \phi_{0, \kappa}(t, x), \dots, \kappa^4 D_t \phi_{m, \kappa}(t, x), \\ &\quad J_x^2 \phi_{0, \kappa}(t, x), \dots, J_x^2 \phi_{m, \kappa}(t, x))). \end{aligned}$$

Recall again that composition by a given smooth function defines smooth maps between Hölder spaces which send bounded sets to bounded sets. Thus, since $R_{1, m}$ and $R_{2, m}$ are themselves smooth functions defined over finite-dimensional manifolds, it follows by the uniform bounds obtained in Lemma 4.3.3 that there exists a positive constant, B_m , such that, for all κ ,

$$\|T_{\kappa} \hat{\Psi}_{\kappa}(\tilde{X}_{m, \kappa}, \tilde{\phi}_{m, \kappa})\|_{0, \alpha, \text{in}, \kappa} \leq B_m \kappa^{m+1},$$

and the result now follows by the implicit function theorem (c.f. [15]). \square

5 - Concentration.

5.1 - Another κ -dependent rescaled time variable. It remains to prove the converse of Theorem 4.4.2. Thus, let F_κ , \mathcal{F}_κ and $\mathcal{E}_\kappa(T^{d+1})$ be defined as in Section 1.3, and, for all small κ , let $[e_{\kappa,t}] : \mathbb{R} \rightarrow \mathcal{E}_\kappa(T^{d+1})$ be an admissible, eternal forced mean curvature flow of bounded type with forcing term F_κ . Given $\kappa > 0$, a smooth curve, $\gamma :]a, b[\rightarrow \mathbb{R}^{d+1}$, and a smooth function, $\psi \in \hat{C}^\infty(]a, b[\times S^d)$, consider the family of embeddings, $\tilde{\Phi}(\kappa, \gamma, \psi) :]a, b[\times S^d \rightarrow \mathbb{R}^{d+1}$, defined by

$$\tilde{\Phi}(\kappa, \gamma, \psi)(r, x) := \gamma(r) + \kappa(1 + \psi(r, x))x. \quad (22)$$

It turns out that the time variable denoted here by r actually lies on a scale intermediate between those of t and s . We therefore denote $t := \kappa^2 r$, and $s := \kappa^{-2} r$, so that $D_t = \kappa^{-2} D_r$ and $D_s = \kappa^2 D_r$. However, we continue to use the κ -dependent Hölder norms, $\|\cdot\|_{k,\alpha,\kappa}$ and $\|\cdot\|_{k,\alpha,\kappa,\text{in}}$, which were defined in Section 4.3 using the operators D_s and δ_s^α .

Lemma 5.1.1

There exists $\kappa_0 > 0$ with the property that for all $\kappa < \kappa_0$, there exists a unique smooth curve, $\gamma_\kappa : \mathbb{R} \rightarrow \mathbb{T}^{d+1}$, and a unique smooth function, $\psi_\kappa \in \hat{C}^\infty(\mathbb{R} \times S^d)$, such that

$$[e_{\kappa,r}] = [\tilde{\Phi}(\kappa, \gamma_\kappa, \psi_\kappa)].$$

Furthermore, for all (k, α) ,

$$\begin{aligned} \kappa^{-1} \|D_s \gamma_\kappa\|_{k,\alpha,\kappa} &\rightarrow 0, \text{ and} \\ \|\psi_\kappa\|_{k,\alpha,\kappa,\text{in}} &\rightarrow 0, \end{aligned}$$

as κ tends to 0.

Proof: We identify the flow, $[e_{\kappa,r}]$, with its lifting in \mathbb{R}^{d+1} . For $\kappa > 0$ and $r_\kappa \in \mathbb{R}$, define the flow, $[\tilde{e}_{\kappa,s}]$, by

$$\tilde{e}_{\kappa,s}(x) := \kappa(e_{\kappa,\kappa^2 s + r_\kappa}(x) - x_\kappa),$$

where x_κ is some interior point of $[e_{\kappa,r_\kappa}]$. Observe that this is an admissible, eternal forced mean curvature flow of bounded type with forcing term

$$1 - \kappa^2 f(x_\kappa + \kappa^{-1} x).$$

By the compactness result of Theorem 1.3.2, there exists $[\tilde{e}_{0,s}]$ towards which $[\tilde{e}_{\kappa,s}]$ sub-converges in the C_{loc}^k -sense for all k as κ tends to 0. Furthermore, $[\tilde{e}_{0,s}]$ is an admissible, eternal forced mean curvature flow of bounded type with constant forcing term 1, so that, by the Hopf Theorem for flows (Theorem 3.1.4), it coincides up to translation with the constant flow

$$\tilde{e}_{0,s}(x) = x.$$

Since this limit is unique, it follows that, up to translation, the entire family, $([\tilde{e}_{\kappa,s}])_{\kappa>0}$, converges in the C_{loc}^k -sense for all k to $[\tilde{e}_{0,s}]$. By the implicit function theorem, there

therefore exists, for all sufficiently small κ , a unique curve, $\tilde{\gamma}_\kappa :]-1, 1[\rightarrow \mathbb{R}^{d+1}$, and a unique smooth function, $\tilde{\psi}_\kappa \in \hat{C}^\infty(]-1, 1[\times S^d)$, such that

$$[\tilde{e}_{\kappa,s}]|_{]-1,1[} = [\tilde{\gamma}_\kappa(s) + (1 + \tilde{\psi}_\kappa(s, x))x].$$

Furthermore, for all (k, α) ,

$$\begin{aligned} \|D_s \tilde{\gamma}_\kappa\|_{k,\alpha,\kappa} &\rightarrow 0 \text{ and} \\ \|\tilde{\psi}_\kappa\|_{k,\alpha,\kappa,\text{in}} &\rightarrow 0, \end{aligned}$$

as κ tends to 0. Rescaling and projecting down to the torus yields a curve $\gamma_\kappa :]r_\kappa - \kappa^2, r_\kappa + \kappa^2[\rightarrow T^{d+1}$ and a smooth function, $\psi_\kappa \in \hat{C}^\infty(]r_\kappa - \kappa^2, r_\kappa + \kappa^2[\times S^d)$, such that

$$[e_\kappa]|_{]r_\kappa - \kappa^2, r_\kappa + \kappa^2[} = [\gamma_\kappa(r) + \kappa(1 + \psi_\kappa(r, x))x].$$

Furthermore, for all (k, α) ,

$$\begin{aligned} \kappa^{-1} \|D_s \gamma_\kappa\|_{k,\alpha,\kappa} &\rightarrow 0, \text{ and} \\ \|\psi_\kappa\|_{k,\alpha,\kappa,\text{in}} &\rightarrow 0, \end{aligned}$$

as κ tends to 0. Finally, since the family $(r_\kappa)_{\kappa>0}$ is arbitrary, this convergence is uniform over $\mathbb{R} \times S^d$, and the result follows. \square

5.2 - Bootstrapping. Consider the function, $\tilde{\Psi} :]0, \infty[\times J \rightarrow \mathbb{R}$, defined such that, for all $\kappa \in]0, \infty[$, for all smooth functions, $\gamma : \mathbb{R} \rightarrow T^{d+1}$ and $\psi : \mathbb{R} \times S^d \rightarrow \mathbb{R}$, and for any point $(r, x) \in \mathbb{R} \times S^d$,

$$\begin{aligned} \tilde{\Psi}(\kappa, J_t^1 \gamma(r), J_{\text{in}}^2 \psi(r, x)) &:= \langle D_r \tilde{\Phi}(\kappa, \gamma, \phi), N(J_x^1 \psi(r, x)) \rangle \\ &+ \frac{1}{\kappa} H(J_x^2 \psi(r, x)) - F_\kappa(\tilde{\Phi}(\kappa, \gamma, \psi(r, x))). \end{aligned} \quad (23)$$

As before, $\tilde{\Psi}$ is a smooth function defined over the finite-dimensional manifold, $]0, \infty[\times J^1$. Likewise, by (5), $\tilde{\Psi}$ has been constructed precisely so that, given $\kappa > 0$, and smooth functions, $\gamma : \mathbb{R} \rightarrow T^{d+1}$ and $\phi : \mathbb{R} \times S^d \rightarrow \mathbb{R}$, the family, $\tilde{\Phi}(\kappa, X, \psi)$, is an eternal forced mean curvature flow with forcing term F_κ if and only if $\tilde{\Psi}(\kappa, J^1 \gamma, J_{\text{in}}^1 \psi)$ vanishes at every point, (r, x) , of $\mathbb{R} \times S^d$.

Lemma 5.2.1

For all (k, α) , there exists $B_{k,\alpha} > 0$ such that, for all $\kappa \in]0, 1]$,

$$\begin{aligned} \|D_r \gamma_\kappa + \kappa^2 \nabla f(\gamma_\kappa(r))\|_{k,\alpha,\kappa} &\leq B_{k,\alpha} \kappa^3, \text{ and} \\ \|\psi_\kappa\|_{k,\alpha,\kappa,\text{in}} &\leq B_{k,\alpha} \kappa^2. \end{aligned}$$

Proof: By (10), (11) and (23),

$$\begin{aligned} \tilde{\Psi} &= \frac{1}{\kappa^2} \langle D_s \gamma_\kappa, x \rangle - \frac{1}{\kappa^2} \langle D_s \gamma_\kappa, \nabla \psi_\kappa \rangle + \frac{1}{\kappa^2} \langle D_s \gamma - \kappa, R_1(x, J_x^1 \psi_\kappa) \rangle \\ &+ \frac{1}{\kappa} D_s \psi_\kappa - \frac{1}{d\kappa} (d + \Delta) \psi_\kappa + \frac{1}{\kappa} R_2(x, D_s \psi_\kappa, J_x^2 \psi_\kappa) \\ &+ \kappa f(\gamma_\kappa) + \kappa^2 \langle \nabla f(\gamma_\kappa), x \rangle + \kappa^2 \psi_\kappa \langle \nabla f(\gamma_\kappa), x \rangle + \kappa^3 R_3(\kappa, x, \gamma_\kappa, \psi_\kappa), \end{aligned}$$

where R_1 , R_2 and R_3 are smooth functions of their arguments and R_1 , R_2 are of order at least 2 in $(D_s\psi_\kappa, J_x^2\psi_\kappa)$. Projecting onto \mathcal{H}_1^\perp yields

$$\begin{aligned} D_s\psi_\kappa + \langle \kappa^{-1}D_s\gamma_\kappa, \nabla\psi_\kappa \rangle + \frac{1}{d}(d + \Delta)\psi_\kappa &= \Pi^\perp R_4(x, \kappa^{-1}D_s\gamma_\kappa, D_s\psi_\kappa, J_s^2\psi_\kappa) \\ &\quad + \kappa^2\Pi^\perp R_5(\kappa, x, \gamma_\kappa, \psi_\kappa), \end{aligned}$$

where R_4 and R_5 are smooth functions of their arguments and R_4 is of order at least 2 in $(D_s\psi_\kappa, J_x^2\psi_\kappa)$. Since the operator $D_s + \frac{1}{d}(d + \Delta)$ maps $\hat{C}^{k+1,\alpha}(\mathbb{R} \times S^d)$ invertibly onto $\hat{C}^{k,\alpha}(\mathbb{R} \times S^d)$, it follows from the estimates already obtained for $D_s\gamma_\kappa$ that the operator on the left hand side is uniformly bounded below for sufficiently small κ . For all (k, α) , there therefore exists $B > 0$ such that

$$\|\psi_\kappa\|_{k,\alpha,\kappa,\text{in}} \leq B\|\psi_\kappa\|_{k,\alpha,\kappa,\text{in}}^2 + B\kappa^2,$$

and since $\|\psi_\kappa\|_{k,\alpha,\kappa,\text{in}}$ tends to 0 as κ tends to 0, the second assertion now follows. Projecting onto \mathcal{H}_1 now yields

$$\begin{aligned} (\text{Id} + M(J_x^1\psi_\kappa))D_t\gamma_\kappa + \kappa^2\nabla f(\gamma_\kappa) &= \frac{1}{\kappa}\Pi R_6(x, D_s\psi_\kappa, J_x^2\psi_\kappa) + \kappa^2\Pi R_7(x, J_x^0\psi_\kappa) \\ &\quad + \kappa^3\Pi R_8(\kappa, x, \gamma_\kappa, \psi_\kappa), \end{aligned}$$

where M , R_6 , R_7 and R_8 are smooth functions of their arguments, R_6 has order at least 2 in $(D_s\psi_\kappa, J_x^2\psi_\kappa)$ and M and R_7 have order 1 in $J_x\phi$, and the first assertion now follows by the bounds obtained on ψ_κ . \square

5.3 - Recovering the Flow. It follows from Theorem 1.3.4 that, as κ tends to 0, the two end-points of the flow, $[e_{\kappa,t}]$, collapse onto critical points of f . In fact (c.f. [18] and [25]), for sufficiently small κ , they coincide with those Alexandrov-embeddings constructed via the elliptic analogue of Theorem 4.4.2. We now show that the entire flow in fact collapses onto a complete gradient flow of f as the parameter, κ , tends to 0.

Lemma 5.3.1

Let M be a compact manifold. If $f : M \rightarrow \mathbb{R}$ is of Morse-Smale type, then there exists $C > 0$ with the following property. If $\epsilon > 0$ and if $\gamma : \mathbb{R} \rightarrow M$ is a C^1 curve such that

$$\begin{aligned} \sup_t \|\dot{\gamma}(t) + \nabla f(\gamma(t))\| &< \epsilon, \\ \limsup_{t \rightarrow -\infty} d(\gamma(t), p_-) &< \epsilon, \text{ and} \\ \limsup_{t \rightarrow +\infty} d(\gamma(t), p_+) &< \epsilon, \end{aligned}$$

where p_- and p_+ are critical points of f and if

$$\text{Index}(p_-) - \text{Index}(p_+) = 1,$$

then there exists a complete integral curve $\gamma_0 : \mathbb{R} \rightarrow M$ of $-\nabla f$ such that

$$\begin{aligned} \lim_{t \rightarrow -\infty} \gamma_0(t) &= p_-, \\ \lim_{t \rightarrow +\infty} \gamma_0(t) &= p_+, \\ \sup_t \|\gamma(t) - \gamma_0(t)\| &< C\epsilon, \text{ and} \\ \int_{-\infty}^{+\infty} \langle \dot{\gamma}_0, \gamma(t) - \gamma_0(t) \rangle dt &= 0. \end{aligned}$$

Remark: Since F is of Morse-Smale type, $\dot{\gamma}_0$, decays exponentially at $\pm\infty$, so that the above integral is always well-defined.

Proof: We first show that for all $\delta > 0$ there exists ϵ_0 such that for $\epsilon < \epsilon_0$, there exists a complete integral curve, $\gamma_0 : \mathbb{R} \rightarrow M$, of $-\nabla f$ such that

$$\begin{aligned} \lim_{t \rightarrow -\infty} \gamma_0(t) &= p_-, \\ \lim_{t \rightarrow +\infty} \gamma_0(t) &= p_+, \text{ and} \\ \sup_t \|\gamma(t) - \gamma_0(t)\| &< \delta. \end{aligned}$$

Indeed, choose $\delta > 0$. By non-degeneracy and compactness, we may suppose that if p is a critical point of f and if $\eta : [0, \infty[\rightarrow M$ is a gradient flow of f starting at some point of $\partial B_{2\delta}(p)$, then

$$\eta([0, \infty[) \cap B_{3\delta}(q) = \emptyset$$

for every other critical point, q , of f such that $\text{Index}(q) \geq \text{Index}(p)$. For every critical point, p , of f , let $S_{2\delta}(p)$ be the intersection of the stable manifold of p with $\partial B_{2\delta}(p)$. By compactness again, for all $\delta' > 0$, there exists $T > 0$ such that if $\eta : [0, \infty[\rightarrow M$ is a gradient flow of f starting at some point of $\partial B_{2\delta}(p)$ such that

$$d(\eta(0), S_{2\delta}(p)) > \delta',$$

then

$$\eta([0, T]) \cap B_\delta(q) \neq \emptyset,$$

for some other critical point q of f , which, in particular, has Morse index strictly less than that of p . However, for sufficiently small ϵ , if $\gamma(t_0) \in \partial B_{2\delta}(p)$ is such that $\dot{\gamma}(t_0)$ points outward from $B_{2\delta}(p)$ or is tangent to $\partial B_{2\delta}(p)$, then, on the one hand, $d(\gamma(t_0), S_{2\delta}(p)) > \delta'$, and, on the other,

$$d(\eta(t), \gamma(t)) < \delta \quad \forall t_0 \leq t \leq t_0 + T,$$

where η is the gradient flow of f such that $\eta(t_0) = \gamma(t_0)$. Consequently, if γ leaves $B_{2\delta}(p)$, then, after a time at most T , it must enter $B_{2\delta}(q)$ for some critical point, q , of f of Morse index strictly less than that of p , after which it cannot return to $B_{2\delta}(p)$.

For ϵ sufficiently small, γ starts inside $B_{2\delta}(p_-)$. By the above discussion, within a time at most T after leaving $B_{2\delta}(p_-)$, the curve, γ , enters $B_{2\delta}(p_+)$ from which it does not leave. Furthermore, there exists a gradient flow η of f from some point of $B_{2\delta}(p_-)$ to some point of $B_{2\delta}(p_+)$ from which γ remains at a distance of at most δ as it travels between these two balls. This proves the assertion.

Now define $X := \gamma(t) - \gamma_0(t)$. In particular,

$$\|X\|_{C^0} < \delta.$$

Since f is of Morse-Smale type, the operator $L := \partial_t - \text{Hess}(f)(\gamma(t))$ is Fredholm and surjective. Furthermore, since $\text{Index}(p_-) - \text{Index}(p_+) = 1$, by the Atiyah-Patodi-Singer index theorem, its kernel is 1-dimensional and is spanned by $\dot{\gamma}_0$. Observe that, upon replacing $\gamma_0(t)$ with $\gamma_0(s+t)$ for a suitable value of s , we may suppose that

$$\int_{-\infty}^{+\infty} \langle \dot{\gamma}_0(t), X(t) \rangle dt = 0,$$

so that X is an element of the orthogonal complement of the kernel of P . However,

$$\begin{aligned} \|LX\|_0 &= \|\dot{X}(t) - \text{Hess}(f)(\gamma_0(t))X(t)\|_0 \\ &= \|\dot{\gamma}(t) - \dot{\gamma}_0(t) - \text{Hess}(f)(\gamma_0(t))X(t)\|_0 \\ &\leq \epsilon + \|\nabla f(\gamma(t)) - \nabla f(\gamma_0(t)) - \text{Hess}(f)(\gamma_0(t))X(t)\|_0 \\ &\leq \epsilon + B_1\|X(t)\|_0^2, \end{aligned}$$

for some constant B_1 , and since the restriction of L to the orthogonal complement of $\dot{\gamma}_0$ is invertible,

$$\|\dot{X} - \text{Hess}(f)(\gamma(t))X\|_0 \geq \frac{1}{B_2}\|X\|_0,$$

for a suitable constant B_2 . Combining these relations yields

$$\|X\|_0(1 - B_1B_2\|X\|_0) \leq B_2\epsilon,$$

and since $\|X\|_0 < \delta$, the result follows. \square

5.4 - Asymptotic series. Finally, recall the function, Ψ , defined in Section 4.1. It follows from Lemmas 5.2.1 and 5.3.1 that for all sufficiently small κ , there exists $X_\kappa \in C_{\text{bdd}}^\infty(\mathbb{R}, \mathbb{R}^{d+1})$ and $\phi_\kappa \in \hat{C}_{\text{bdd}}^\infty(\mathbb{R} \times S^d)$ such that

$$[e_{\kappa,t}] = \Psi(\kappa, X_\kappa, \phi_\kappa).$$

Furthermore, for all (k, α) , there exists $B_{k,\alpha} > 0$ such that, for all $\kappa \in]0, 1]$,

$$\begin{aligned} \|X_\kappa\|_{k,\alpha,\kappa} &\leq B_{k,\alpha}, \\ \|D_t X_\kappa\|_{k,\alpha,\kappa} &\leq B_{k,\alpha}, \text{ and} \\ \|\phi_\kappa\|_{k,\alpha,\kappa,\text{in}} &\leq B_{k,\alpha}. \end{aligned}$$

Consider now the formal solutions, $\sum_{k=0}^\infty \kappa^k X_{m,\kappa}$ and $\sum_{k=0}^\infty \kappa^k \phi_{m,\kappa}$, constructed in Lemma 4.2.1.

Lemma 5.4.1

The family (X_κ, ϕ_κ) satisfies

$$X_\kappa \sim \sum_{k=0}^{\infty} \kappa^k X_{k,\kappa}, \text{ and}$$

$$\phi_\kappa \sim \sum_{k=0}^{\infty} \kappa^k \phi_{k,\kappa}.$$

Proof: This is proven by induction. Indeed, denote

$$\tilde{X}_\kappa := \kappa^{-(m+1)} \left(X_\kappa - \sum_{k=0}^m \kappa^k X_{k,\kappa} \right), \text{ and}$$

$$\tilde{\phi}_\kappa := \kappa^{-(m+1)} \left(\phi_\kappa - \sum_{k=0}^m \kappa^k \phi_{k,\kappa} \right).$$

Suppose, by induction, that, for all (k, α) , there exists $B_{k,\alpha}$ such that, for all $\kappa \in]0, 1]$,

$$\|\tilde{X}_\kappa\|_{k,\alpha,\kappa} \leq B_{k,\alpha},$$

$$\|D_t \tilde{X}_\kappa\|_{k,\alpha,\kappa} \leq B_{k,\alpha}, \text{ and}$$

$$\|\tilde{\phi}_\kappa\|_{k,\alpha,\kappa} \leq B_{k,\alpha}.$$

Consider the asymptotic formula, (15), for Ψ , substituting $X_k = X_{k,\kappa}$, for $0 \leq k \leq m$, $X_{m+1} = \tilde{X}_\kappa$, and $X_k = 0$, for $k \geq m+2$; and substituting $\phi_k = \phi_{k,\kappa}$, for $0 \leq k \leq m$, $\phi_{m+1} = \tilde{\phi}_\kappa$ and $\phi_k = 0$, for $k \geq m+2$. Projecting the $(m+2)$ 'nd order term orthogonally onto $\hat{\mathcal{H}}$, we see that there exists a smooth function, R_1 , such that

$$P_\kappa \tilde{\phi}_\kappa = -\langle \nabla f(\gamma(t)), X_{m,\kappa} \rangle - \langle \text{Hess}(f)(\gamma(t)) X_{m-1,\kappa}, X_{0,\kappa} \rangle$$

$$- \Pi^\perp \left(\Psi_{m+1}(\gamma, D_t \gamma, x, J_t^1 X_{0,\kappa}, \dots, J_t^1 X_{m-2,\kappa}, \right.$$

$$\left. \kappa^4 D_t \phi_{0,\kappa}, \dots, \kappa^4 D_t \phi_{m-1,\kappa}, J_x^2 \phi_{0,\kappa}, \dots, J_x^2 \phi_{m-1,\kappa} \right)$$

$$- \kappa \Pi^\perp \left(R_1(\gamma, D_t \gamma, x, J_t^1 X_{0,\kappa}, \dots, J_t^1 X_{m,\kappa}, J_t^1 \tilde{X}_\kappa \right.$$

$$\left. \kappa^4 D_t \phi_{0,\kappa}, \dots, \kappa^4 D_t \phi_{m,\kappa}, \kappa^4 D_t \tilde{\phi}_\kappa, J_x^2 \phi_{0,\kappa}, \dots, J_x^2 \phi_{m,\kappa}, J_x^2 \tilde{\phi}_\kappa \right).$$

However, for all (k, α) , the operator P_κ defines an invertible linear map from $\hat{C}_{\text{in}}^{k+1,\alpha}(\mathbb{R} \times S^d)$ into $\hat{C}_{\text{in}}^{k,\alpha}(\mathbb{R} \times S^d)$. It therefore follows from the definition, (18), of $\phi_{k+1,\kappa}$ that there exists a constant, $B > 0$, such that, for all $\kappa \in]0, 1]$,

$$\|\tilde{\phi}_\kappa - \phi_{k+1,\kappa}\|_{k,\alpha,\kappa} \leq B\kappa.$$

Denote now

$$\tilde{\phi}'_\kappa := \kappa^{-1} \left(\tilde{\phi}_\kappa - \phi_{k+1,\kappa} \right).$$

Consider again the asymptotic formula, (15), for Ψ , substituting again $X_k = X_{k,\kappa}$, for $0 \leq k \leq m$, $X_{m+1} = \tilde{X}_\kappa$ and $X_k = 0$, for $k \geq m+2$; but substituting this time $\phi_k = \phi_{k,\kappa}$, for $0 \leq k \leq m+1$, $\phi_{m+2} = \tilde{\phi}'_\kappa$ and $\phi_k = 0$, for $k \geq m+3$. Projecting the $(m+4)$ 'th order term orthogonally onto \mathcal{H}_1 , we see that there exists a smooth function, R_2 , such that

$$\begin{aligned} L\tilde{X} = & -\Pi(\Psi_{m+1}(\gamma, D_t\gamma, x, J_t^1 X_{0,\kappa}, \dots, J_t^1 X_{m,\kappa}, \\ & \kappa^4 D_t \phi_{0,\kappa}, \dots, \kappa^4 D_t \phi_{m+1,\kappa}, J_x^2 \phi_{0,\kappa}, \dots, J_x^2 \phi_{m+1,\kappa})) \\ & - \kappa \Pi(R_2(\gamma, D_t\gamma, x, J_t^1 X_{0,\kappa}, \dots, J_t^1 X_{m,\kappa}, J_t^1 \tilde{X}_\kappa \\ & \kappa^4 D_t \phi_{0,\kappa}, \dots, \kappa^4 D_t \phi_{m+1,\kappa}, \kappa^4 D_t \tilde{\phi}'_\kappa, J_x^2 \phi_{0,\kappa}, \dots, J_x^2 \phi_{m+1,\kappa}, J_x^2 \tilde{\phi}'_\kappa)), \end{aligned}$$

However, by Lemma 4.3.2, for all (k, α) , the norm of K is uniformly bounded independent of $\kappa \in]0, 1]$. It therefore follows by the definition, (19), of $X_{k+1,\kappa}$ that there exists a constant, $B > 0$, such that

$$\|\tilde{X} - X_{k+1,\kappa}\|_{k,\alpha} \leq B\kappa.$$

The result now follows by induction. \square

A - The weakly smooth category.

Let M be a $(d+1)$ -dimensional riemannian manifold, and let $\mathcal{E}(M)$ be defined as in the introduction. Charts of $\mathcal{E}(M)$ are constructed as follows. For $[e] \in \mathcal{E}(M)$, let $N_e : S^d \rightarrow TM$ denote the unit, normal vector field over e which is compatible with the orientation, and consider the map, $E_e : S^d \times \mathbb{R} \rightarrow M$, given by

$$E_e(x, t) := \text{Exp}_{e(x)}(tN_e(x)),$$

where Exp is the exponential map of M . Let $\epsilon_e > 0$ be such that the restriction of E_e to $S^d \times]-\epsilon_e, \epsilon_e[$ is an immersion (c.f. Proposition 3.2 of [11]), and define the open subset, \mathcal{U}_e , of $C^\infty(S^d)$ and the mapping $\hat{\Phi}_e : \mathcal{E}_e \rightarrow C^\infty(S^d, M)$ by

$$\begin{aligned} \mathcal{U}_e &:= \{\phi \in C^\infty(S^d) \mid \|\phi\|_{C^0} < \epsilon_e\}, \text{ and} \\ \hat{\Phi}_e(\phi)(x) &:= E_e(x, \phi(x)). \end{aligned}$$

Upon reducing ϵ_e if necessary, we may suppose that $\hat{\Phi}_e(\phi)$ is an Alexandrov-embedding for all $\phi \in \mathcal{U}_e$ so that $\hat{\Phi}_e$ projects down to a mapping, Φ_e , which is in fact a homeomorphism from \mathcal{U}_e into an open subset, \mathcal{V}_e , of $\mathcal{E}(M)$ (c.f. Propositions 3.3 and 3.4 of [11]). The triplet $(\Phi_e, \mathcal{U}_e, \mathcal{V}_e)$ is called the **graph chart** of $\mathcal{E}(M)$ about e . The set of all graph charts constitutes an atlas of $\mathcal{E}(M)$ all of whose transition maps are homeomorphisms, thereby furnishing $\mathcal{E}(M)$ with the structure of a topological manifold modelled on $C^\infty(S^d)$.

This atlas in fact furnishes $\mathcal{E}(M)$ with the structure of a smooth-tame Frechet manifold (c.f. [8] for a splendid introduction to this theory). However, since this framework is highly technical, we prefer to avoid it whenever possible. It is for this reason that we introduce the following formalism of weakly smooth manifolds which, despite its simplicity, possesses all the structure required for the development of the formal parts of the theory. Thus, consider first an open subset, \mathcal{U} , of $C^\infty(S^d)$. Given a finite-dimensional manifold, X , and

a function, $c : X \rightarrow \mathcal{U}$, the function, $\tilde{c} : X \times S^d \rightarrow M$, is defined by $\tilde{c}(x, y) := c(x)(y)$. The function, c , is then said to be **strongly smooth** whenever the function, \tilde{c} , is smooth. Now consider another open subset, \mathcal{U}' , of $C^\infty(S^d)$. A function, $\Phi : \mathcal{U} \rightarrow \mathcal{U}'$, is said to be **weakly smooth** whenever composition by Φ sends strongly smooth functions continuously into strongly smooth functions. Observe, in particular, that every weakly smooth function is also continuous. It is now a straightforward matter to show that the transition maps between graph charts of $\mathcal{E}(M)$ are weakly smooth, so that $\mathcal{E}(M)$ carries the structure of a weakly smooth manifold. In particular, a function $c : X \rightarrow \mathcal{E}(M)$ is now said to be **strongly smooth** whenever it is strongly smooth in every graph chart.

The formalism of weakly smooth manifolds possesses considerable structure. For example, tangent vectors are well-defined, as is the tangent bundle, which is also a weakly smooth manifold, and so on. The main important result not admitted by this formalism is the inverse function theorem. However, this presents no problem in the present context, since ellipticity and hypo-ellipticity allow us to work locally in every graph chart as if it were a Hölder space, where the inverse function theorem for Banach manifolds can then be applied. For a more thorough discussion, we refer the reader to [14].

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